Isbell conjugacy and the reflexive completion

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Abstract

The reflexive completion of a category consists of the **Set**-valued functors on it that are canonically isomorphic to their double conjugate. After reviewing both this construction and Isbell conjugacy itself, we give new examples and revisit Isbell's main results from 1960 in a modern categorical context. We establish the sense in which reflexive completion is functorial, and find conditions under which two categories have equivalent reflexive completions. We describe the relationship between the reflexive and Cauchy completions, determine exactly which limits and colimits exist in an arbitrary reflexive completion, and make precise the sense in which the reflexive completion of a category is the intersection of the categories of covariant and contravariant functors on it.

Contents

1 Introduction

Isbell conjugacy inhabits the same basic level of category theory as the Yoneda lemma, springing from the most primitive concepts of the subject: category, functor and natural transformation. It can be understood as follows.

Let $\mathcal{A}$ be a small category. Any functor $X\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ gives rise to a new functor $X'\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$X'(a) = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](\mathcal{A}(-,a), X),$$

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and so, in principle, an infinite sequence $X, X', X'', \dots$ of functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. Of course, they are all canonically isomorphic, by the Yoneda lemma. But X also gives rise to a functor $X^\vee \colon \mathcal{A} \to \mathbf{Set}$, its *Isbell conjugate*, defined by

$$X^\vee(a) = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](X, \mathcal{A}(-, a)). \tag{1}$$

The same construction with $\mathcal{A}$ in place of $\mathcal{A}^{\mathrm{op}}$ produces from $X^\vee$ a further functor $X^{\vee\vee} \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$, and so on, giving an infinite sequence $X, X^\vee, X^{\vee\vee}, \dots$ of functors on $\mathcal{A}$ with alternating variances. Although it makes no sense to ask whether $X^\vee$ is isomorphic to X (their types being different), one can ask whether $X^{\vee\vee} \cong X$. This is false in general. Thus, there is nontrivial structure.

The conjugacy operations define an adjunction between $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ and $[\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$, so that

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](X, Y^\vee) \cong [\mathcal{A}, \mathbf{Set}](Y, X^\vee)$$

naturally in $X \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ and $Y \colon \mathcal{A} \to \mathbf{Set}$. The unit and counit of the adjunction are canonical maps $X \to X^{\vee\vee}$ and $Y \to Y^{\vee\vee}$, and a covariant or contravariant functor on $\mathcal{A}$ is said to be *reflexive* if the canonical map to its double conjugate is an isomorphism.

The *reflexive completion* $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ is the category of reflexive functors on $\mathcal{A}$ (covariant or contravariant; it makes no difference). Put another way, $\mathcal{R}(\mathcal{A})$ is the invariant part of the conjugacy adjunction. It contains $\mathcal{A}$, since representables are reflexive. Its properties are the main subject of this work.

The reflexive completion is very natural category-theoretically, but categories of reflexive objects also appear in other parts of mathematics. That is, there are many notions of duality in mathematics, in most instances there is a canonical map $\eta_X \colon X \to X^{**}$ from each object X to its double dual, and special attention is paid to those X for which η_X is an isomorphism. For example, in linear algebra, the vector spaces X with this property are the finite-dimensional ones, and in functional analysis, there is a highly developed theory of reflexivity for Banach spaces and topological vector spaces.

Content of the paper We begin with the definition of conjugacy on *small* categories, giving several characterizations of the conjugacy operations and many examples (Sections 2 and 3). Defining conjugacy on an arbitrary category is more delicate, and we review and use the notion of small functor (Section 4). This allows us to state the definition of the reflexive completion of an arbitrary category, and again, we give many examples (Sections 5 and 6).

Up to here, there are no substantial theorems, but the examples provide some surprises. For instance, the reflexive completion of a nontrivial group is simply the group with initial and terminal objects adjoined—except when the group is of order 2, in which case it is something more complicated (for reasons related to the fact that $2 + 2 = 2 \times 2$; see Example 6.5). There is also a finite monoid whose reflexive completion is not even small, a fact due to Isbell (Examples 6.8 and 8.7). Other examples involve the Dedekind–MacNeille completion of an ordered set (Examples 6.10 and 6.11) and the tight span of a metric space (at the end of Section 6).

The second half of the paper develops the theory, as follows.

Section 7 collects necessary results on dense and adequate functors. (See Definition 7.10 and Remark 7.11 for this terminology.) Many of them are standard, but we address points about set-theoretic size that do not seem to have

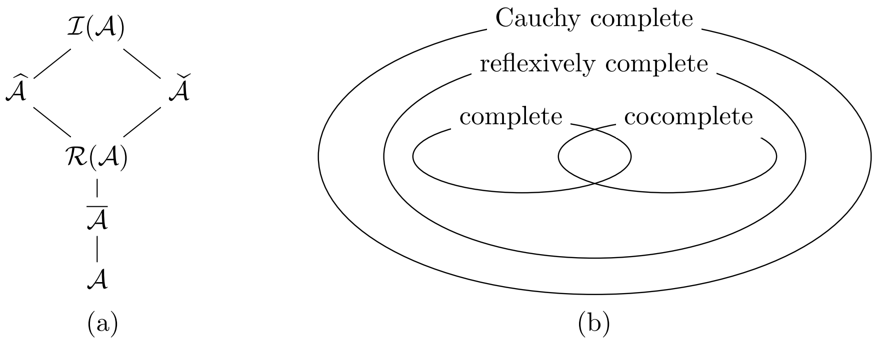

Figure 1: (a) Completions of a category $\mathcal{A}$: the Cauchy completion $\overline{\mathcal{A}}$, reflexive completion $\mathcal{R}(\mathcal{A})$, free completions $\widehat{\mathcal{A}}$ and $\check{\mathcal{A}}$ with respect to small colimits and small limits, and Isbell envelope $\mathcal{I}(\mathcal{A})$; (b) classes of complete categories.

previously been considered. Using the results of Section 7, we give a unique characterization of the reflexive completion that sharpens a result of Isbell's (Theorem 8.4).

Reflexive completion is functorial (Section 9), but only with respect to a very limited class of functors: the small-adequate ones. It is often the case that the functor $\mathcal{R}(F)\colon \mathcal{R}(\mathcal{B}) \to \mathcal{R}(\mathcal{A})$ induced by a functor $F\colon \mathcal{A} \to \mathcal{B}$ is an equivalence. For example, using work of Day and Lack on small functors together with the results on size just mentioned, we show that $\mathcal{R}(F)$ is always an equivalence if $\mathcal{B}$ is either small or both complete and cocomplete (Corollary 9.8).

Our study of functoriality naturally recovers Isbell's result that reflexive completion is idempotent: $\mathcal{R}(\mathcal{R}(\mathcal{A})) \simeq \mathcal{R}(\mathcal{A})$. A category is *reflexively complete* if it is the reflexive completion of some category, or equivalently if every reflexive functor on it is representable.

Reflexive completion has certain formal resemblances to Cauchy completion, but the reflexive completion is typically bigger (Figure 1). The relationship is analysed in Section 10.

A reflexively complete category has absolute (co)limits, and if it is the reflexive completion of a *small* category then it has initial and terminal objects too, but these are all the limits and colimits that it generally has (Section 11). The case of ordered sets, where the reflexive (Dedekind–MacNeille) completion has all (co)limits, is atypical. On the other hand, it *is* true that a complete or cocomplete category is reflexively complete (Figure 1).

Informally, one can understand $\mathcal{R}(\mathcal{A})$ as the intersection $\widehat{\mathcal{A}} \cap \check{\mathcal{A}}$, where $\widehat{\mathcal{A}}$ and $\check{\mathcal{A}}$ are the free completions of $\mathcal{A}$ under small colimits and small limits. (If $\mathcal{A}$ is small then $\widehat{\mathcal{A}} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ and $\check{\mathcal{A}} = [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$.) Section 12 formalizes this idea, reviewing the definition of the Isbell envelope $\mathcal{I}(\mathcal{A})$ of a category and proving that the square in Figure 1(a) is a pullback in the bicategorical sense.

We work with categories enriched over a suitable monoidal category $\mathcal{V}$ in Sections 2–6, then restrict to $\mathcal{V} = \mathbf{Set}$ from Section 7. While some of the later results are particular to $\mathcal{V} = \mathbf{Set}$ (such as Theorem 11.6 on limits), others can be generalized to any $\mathcal{V}$. To avoid complicating the presentation, we have not specified exactly which results generalize, but we have tried to choose proofs that make any generalization transparent.

Relationship to Isbell's paper Although there are many new results in this work, some parts are accounts of results first proved in Isbell's remarkable paper [9], and the reader may ask what we bring that Isbell did not. There are several answers.

First, Isbell's paper was extraordinarily early. He submitted it in mid-1959, only the year after the publication of Kan's paper introducing adjoint functors. What we now know about category theory can be used to give shape to Isbell's original arguments. In Grothendieck's metaphor [19], the rising sea of general category theory has made the hammer and chisel unnecessary.

Second, Isbell worked only with full subcategories, where we use arbitrary functors. It is true that we will often need to assume our functors to be full and faithful, so that up to equivalence, they are indeed inclusions of full subcategories. Nevertheless, the functor-based approach has the benefits of being equivalence-invariant and of revealing exactly where the full and faithful hypothesis is needed. Ulmer emphasized that many naturally occurring dense functors are not full and faithful (Example 7.3), and the theory of dense and adequate functors should be developed as far as possible without that assumption.

Third, we modernize some aspects, including the treatment of set-theoretic size. Isbell used a size constraint on **Set**-valued functors that he called properness and Freyd later called pettiness (Remark 4.5). It now seems clear that the most natural such notion is that of small functor, which extends smoothly to the enriched context and is what we use here.

Finally, Isbell simply omitted several proofs; we provide them.

Terminology Isbell conjugacy has sometimes been called Isbell duality (as in Di Liberti [4]), but that term has also been used for a different purpose entirely (as in Barr, Kennison and Raphael [1]). Reflexive completion has also been studied under the name of Isbell completion (as in Willerton [21]).

Conventions Usually, and always in declarations such as 'let $\mathcal{A}$ be a category', the word 'category' means *locally small* category. However, we will sometimes form categories such as $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ that are not locally small. For us, the words **small** and **large** refer to sets and proper classes.

The symbol $\times$ denotes both product and copower, so that when S is a set and a is an object of some category, $S \times a = \coprod_{s \in S} a$.

A category is (co)complete when it admits small (co)limits.

2 Conjugacy for small categories

Certain aspects of conjugacy are simpler for small categories. In this section, we review several descriptions and characterizations of conjugacy on small categories, all previously known. Our categories will be enriched in a complete and cocomplete symmetric monoidal closed category $\mathcal{V}$. Henceforth, we will usually abbreviate '$\mathcal{V}$-category' to 'category', and similarly for functors, adjunctions, etc.; all are understood to be $\mathcal{V}$-enriched.

Let $\mathcal{A}$ be a small category. The **(Isbell) conjugate** of a functor $X \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is the functor $X^\vee \colon \mathcal{A} \to \mathcal{V}$ defined by

$$X^\vee(a) = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](X, \mathcal{A}(-, a))$$

($a \in \mathcal{A}$). With $\mathcal{A}^{\mathrm{op}}$ in place of $\mathcal{A}$, this means that the conjugate of a functor $Y\colon \mathcal{A} \to \mathcal{V}$ is the functor $Y^\vee\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ defined by

$$Y^\vee(a) = [\mathcal{A}, \mathcal{V}](Y, \mathcal{A}(a, -)).$$

Remark 2.1 Every functor Z from a small category to $\mathcal{V}$ has a single, unambiguous, conjugate $Z^\vee$, which is the same whether Z is regarded as a covariant functor on its domain or a contravariant functor on the opposite of its domain.

Conjugacy defines a pair of functors

$$[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \underset{\vee}{\overset{\vee}{\rightleftarrows}} [\mathcal{A}, \mathcal{V}]^{\mathrm{op}} . \tag{2}$$

Given $X\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ and $Y\colon \mathcal{A} \to \mathcal{V}$, define

$$\begin{array}{rccc} X \boxtimes Y\colon & \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} & \to & \mathcal{V} \\ & (a, b) & \mapsto & X(a) \otimes Y(b). \end{array} \tag{3}$$

One verifies that

$$[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](X, Y^\vee) \cong [\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}](X \boxtimes Y, \mathrm{Hom}_{\mathcal{A}}) \cong [\mathcal{A}, \mathcal{V}](Y, X^\vee) \tag{4}$$

naturally in X and Y. In particular, the conjugacy functors (2) define a contravariant adjunction on the right.

Evidently

$$\mathcal{A}(-, a)^\vee \cong \mathcal{A}(a, -), \qquad \mathcal{A}(a, -)^\vee \cong \mathcal{A}(-, a)$$

naturally in $a \in \mathcal{A}$. Thus, both triangles in the diagram

$$\begin{tikzcd} {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \arrow[rr, bend left=10, "\vee"] \arrow[rr, phantom, "\perp"] & & {[\mathcal{A}, \mathcal{V}]^{\mathrm{op}}} \arrow[ll, bend left=10, "\vee"] \\ & \mathcal{A} \arrow[ul, "H_\bullet"] \arrow[ur, "H^\bullet"'] & \end{tikzcd} \tag{5}$$

commute, where $H_\bullet$ and $H^\bullet$ are the two Yoneda embeddings. This property characterizes conjugacy:

Lemma 2.2 *Isbell conjugacy is the unique adjunction such that both triangles in (5) commute up to isomorphism.*

Proof Let $P\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}]^{\mathrm{op}}$ be a left adjoint satisfying $P \circ H_\bullet \cong H^\bullet$. By hypothesis, $P(X) \cong X^\vee$ when X is representable. But every object of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is a small colimit of representables (as $\mathcal{A}$ is small), and both P and $(\ )^\vee$ preserve colimits (being left adjoints), so $P \cong (\ )^\vee$. $\square$

Conjugacy can also be described as a nerve-realization adjunction. Any functor $F\colon \mathcal{A} \to \mathcal{E}$ induces a **nerve functor**

$$\begin{array}{rccc} N_F\colon & \mathcal{E} & \to & [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \\ & E & \mapsto & \mathcal{A}(F-, E). \end{array}$$

When $\mathcal{E}$ is cocomplete, the nerve functor has a left adjoint, sometimes called the realization functor of F (after the case where F is the standard embedding of the simplex category Δ into **Top**). It is the left Kan extension of F along the Yoneda embedding $H_\bullet$.

Taking F to be $H^\bullet \colon \mathcal{A} \to [\mathcal{A}, \mathcal{V}]^{\mathrm{op}}$, we thus obtain a pair of adjoint functors between $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and $[\mathcal{A}, \mathcal{V}]^{\mathrm{op}}$. This is the conjugacy adjunction. For example, in diagram (5), the functor $(\ )^\vee \colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}]^{\mathrm{op}}$ is the left Kan extension of $H^\bullet$ along $H_\bullet$.

Yet another derivation of conjugacy uses profunctors. Our convention is that for small categories $\mathcal{A}$ and $\mathcal{B}$, a **profunctor** $\mathcal{B} \nrightarrow \mathcal{A}$ is a functor $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B} \to \mathcal{V}$, and the composite of profunctors $Q \colon \mathcal{C} \nrightarrow \mathcal{B}$ and $P \colon \mathcal{B} \nrightarrow \mathcal{A}$ is denoted by $P \odot Q \colon \mathcal{C} \nrightarrow \mathcal{A}$.

The operation of composition with a profunctor, on either the left or the right, has a right adjoint. Indeed, given profunctors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q", "\shortmid" marking] \arrow[rr, bend right, "R"', "\shortmid" marking] & \mathcal{B} \arrow[r, "P", "\shortmid" marking] & \mathcal{A}, \end{tikzcd}$$

there are profunctors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "{[P,R]_>}", "\shortmid" marking] & \mathcal{B} \arrow[r, "{[Q,R]_<}", "\shortmid" marking] & \mathcal{A} \end{tikzcd}$$

defined by

$$\begin{aligned} [P, R]_>(b, c) &= [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](P(-, b), R(-, c)), \\ [Q, R]_<(a, b) &= [\mathcal{C}, \mathcal{V}](Q(b, -), R(a, -)), \end{aligned}$$

which satisfy the adjoint correspondences

$$\frac{\dfrac{P \to [Q, R]_<}{P \odot Q \to R}}{Q \to [P, R]_>} \tag{6}$$

Now take $\mathcal{B}$ to be the unit $\mathcal{V}$-category $\mathcal{I}$, with $\mathcal{C} = \mathcal{A}$ and $R = \mathrm{Hom}_{\mathcal{A}}$. The profunctors P and Q are functors $X \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ and $Y \colon \mathcal{A} \to \mathcal{V}$, respectively. Then $[P, R]_> = X^\vee$ and $[Q, R]_< = Y^\vee$, while $P \odot Q = X \boxtimes Y$, and the general adjointness relations (6) reduce to the conjugacy relations (4).

Finally, conjugates can be described as Kan extensions or lifts in the bicategory $\mathcal{V}$-**Prof** of $\mathcal{V}$-profunctors. Let $X \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$. There is a canonical natural transformation

$$\begin{tikzcd} \mathcal{A} \arrow[r, "X^\vee", "\shortmid" marking] \arrow[dr, "\mathrm{Hom}_{\mathcal{A}}"', "\shortmid" marking] & \mathbf{1} \arrow[d, "X", "\shortmid" marking] \arrow[dl, phantom, "\Swarrow \varepsilon_X"] \\ & \mathcal{A} \end{tikzcd} \tag{7}$$

whose (a, b)-component

$$X(a) \otimes [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](X, \mathcal{A}(-, b)) \to \mathcal{A}(a, b)$$

is defined in the case $\mathcal{V} = \mathbf{Set}$ by $(x, \xi) \mapsto \xi_b(x)$, and by the obvious generalization for arbitrary $\mathcal{V}$. Equivalently, using the second of the isomorphisms (4), ε_X is the map $X \boxtimes X^\vee \to \mathrm{Hom}_\mathcal{A}$ corresponding to the identity on $X^\vee$.

The result is that ε_X exhibits $X^\vee$ as the right Kan lift of $\mathrm{Hom}_\mathcal{A}$ through X in $\mathcal{V}$-**Prof**. (That is, the pair $(X^\vee, \varepsilon_X)$ is terminal of its type.) This follows from the second adjointness relation in (6) on taking $P = X$ and $R = \mathrm{Hom}_\mathcal{A}$.

Dually, for $Y \colon \mathcal{A} \to \mathcal{V}$, a similarly defined transformation $\varepsilon_Y : Y^\vee \odot Y \to \mathrm{Hom}_\mathcal{A}$ exhibits $Y^\vee$ as the right Kan extension of $\mathrm{Hom}_\mathcal{A}$ along Y in $\mathcal{V}$-**Prof**.

3 Examples of conjugacy

We list some examples of conjugacy, beginning with unenriched categories.

Example 3.1 Let $\mathcal{A}$ be a small discrete category, $Y \colon \mathcal{A} \to \mathbf{Set}$, and $a \in \mathcal{A}$. Then

$$Y^\vee(a) = \begin{cases} 1 & \text{if } Y(b) = \varnothing \text{ for all } b \neq a \\ \varnothing & \text{otherwise.} \end{cases}$$

Thus, writing

$$\operatorname{supp} Y = \{b \in \mathcal{A} : Y(b) \text{ is nonempty}\}, \tag{8}$$

we have

$$Y^\vee \cong \begin{cases} 1 & \text{if } Y \cong 0 \\ \mathcal{A}(-, a) & \text{if } \operatorname{supp} Y = \{a\} \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.2 Let G be a group, seen as a one-object category. A functor $X \colon G^{\mathrm{op}} \to \mathbf{Set}$ is a right G-set, and the unique representable such functor is G_r, the set G acted on by G by right multiplication. Thus, $X^\vee$ is the left G-set of G-equivariant maps $G \to G_r$. We now compute $X^\vee$ explicitly.

First suppose that the G-set X is nonempty, transitive and **free** (for $g \in G$, if $xg = x$ for some x then $g = 1$). Then $X \cong G_r$, so $X^\vee$ is isomorphic to G_ℓ, the set G acted on by the group G by left multiplication.

Next suppose that X is nonempty and transitive but not free. Choose $x \in X$ and $1 \neq g \in G$ such that $xg = x$. Any equivariant $\alpha \colon X \to G_r$ satisfies $\alpha(x) = \alpha(xg) = \alpha(x)g$, a contradiction since $g \neq 1$. Hence $X^\vee = \varnothing$.

Finally, take an arbitrary G-set X. It is a coproduct $\sum_{i \in I} X_i$ of nonempty transitive G-sets, so by adjointness, $X^\vee = \prod_{i \in I} X_i^\vee$. By the previous paragraph, $X^\vee$ is empty unless every orbit X_i is free, or equivalently unless X is free. If X is free then X is the copower $I \times G_r$ and $X^\vee$ is the power G_ℓ^I. But $I \cong X/G$, so

$$X^\vee \cong \begin{cases} G_\ell^{X/G} & \text{if } X \text{ is free} \\ \varnothing & \text{otherwise.} \end{cases}$$

Example 3.3 Let A be a partially ordered set regarded as a category, and let $X \colon A^{\mathrm{op}} \to \mathbf{Set}$. The set $\operatorname{supp} X \subseteq A$ (equation (8)) is downwards closed, and when $X = A(-, a)$, it is $\downarrow a = \{b \in A : b \leq a\}$. Now

$$X^\vee(a) \cong \begin{cases} 1 & \text{if } \operatorname{supp} X \subseteq \downarrow a \\ \varnothing & \text{otherwise.} \end{cases}$$

Of course, the dual result also holds, involving $\uparrow a = \{b \in A : b \geq a\}$.

A **Set**-valued functor on a category is **subterminal** if it is a subobject of the terminal functor, or equivalently if all of its values are empty or singletons. Subterminal functors $A \to \mathbf{Set}$ correspond via supp to upwards closed subsets of A. The conjugate of any functor $X\colon A^{\mathrm{op}} \to \mathbf{Set}$ is subterminal, corresponding to the upwards closed set of upper bounds of $\operatorname{supp} X$ in A.

Example 3.4 Write $\mathbf{2} = (0 \to 1)$ with min as monoidal structure. A small $\mathbf{2}$-category A is a partially ordered set (up to equivalence), and a $\mathbf{2}$-functor $X\colon A^{\mathrm{op}} \to \mathbf{2}$ amounts to a downwards closed subset of A, namely, $\{a \in A : X(a) = 1\}$. Dually, a $\mathbf{2}$-functor $A \to \mathbf{2}$ is an upwards closed subset of A.

From this perspective, the conjugacy adjunction is as follows: for a downwards closed set $X \subseteq A$, the upwards closed set $X^\vee$ is the set of upper bounds of X, and dually.

Example 3.5 Write **Ab** for the category of abelian groups. A one-object **Ab**-category R is a ring, and an **Ab**-functor $R^{\mathrm{op}} \to \mathbf{Ab}$ is a right R-module. The unique representable on R is R_r, the abelian group R regarded as a right R-module. Thus, the conjugate of a right module M is

$$M^\vee = \mathbf{Mod}_R(M, R_r)$$

with the left module structure induced by the left action of R on itself. When R is a field, $M^\vee$ is the dual of the vector space M.

Example 3.6 Consider the ordered set $([0,\infty], \geq)$ with its additive monoidal structure. This is a monoidal closed category, the internal hom $[x, y]$ being the truncated difference

$$y \mathbin{\dot{-}} x = \max\{y - x, 0\}.$$

Lawvere [16] famously observed that a $[0,\infty]$-category is a generalized metric space, 'generalized' in that distances need not be symmetric or finite, and distinct points can be distance 0 apart.

Let $A = (A, d)$ be a generalized metric space. A $[0,\infty]$-functor $A^{\mathrm{op}} \to [0,\infty]$ is a function $f\colon A \to [0,\infty]$ such that

$$f(a) \mathbin{\dot{-}} f(b) \leq d(a, b)$$

for all $a, b \in A$. Its conjugate $f^\vee\colon A \to [0,\infty]$ is defined by

$$f^\vee(a) = \sup_{b \in A} \bigl(d(b, a) \mathbin{\dot{-}} f(b)\bigr).$$

4 Conjugacy for general categories

To define conjugacy on a general category requires more delicacy than on a small category. The reader who wants to get on to the reflexive completion can ignore this section for now. However, because of the phenomenon noted in Example 6.8, the *theory* of the reflexive completion ultimately requires this more general definition of conjugacy: it is not possible to confine oneself to small categories only.

The following example shows that the definition of conjugacy for small categories cannot be extended verbatim to large categories.

Example 4.1 Let $\mathcal{C}$ be a proper class. Let $\mathcal{A}$ be the category obtained by adjoining to the discrete category $\mathcal{C}$ a further object z and maps $p_c^0, p_c^1 \colon z \to c$ for each $c \in \mathcal{C}$. Let $Y \colon \mathcal{A} \to \mathbf{Set}$ be the functor defined by

$$Y(a) = \begin{cases} 1 & \text{if } a \in \mathcal{C} \\ \varnothing & \text{if } a = z. \end{cases}$$

A natural transformation $Y \to \mathcal{A}(z, -)$ is a choice of element of $\{p_c^0, p_c^1\}$ for each $c \in \mathcal{C}$. There is a proper class of such transformations, so there is no **Set**-valued functor $Y^\vee \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ defined by $Y^\vee(a) = [\mathcal{A}, \mathbf{Set}](Y, \mathcal{A}(a, -))$.

Since not every functor has a conjugate, we restrict ourselves to a class of functors that do. These are the small functors introduced by Ulmer ([20], Remark 2.29). We briefly review them now, referring to Day and Lack [3] for details.

Again we work over a complete and cocomplete symmetric monoidal closed category $\mathcal{V}$, understanding all categories, functors, etc., to be $\mathcal{V}$-enriched.

For a category $\mathcal{A}$, a functor $\mathcal{A} \to \mathcal{V}$ is **small** if it can expressed as a small colimit of representables, or equivalently if it is the left Kan extension of its restriction to some small full subcategory of $\mathcal{A}$, or equivalently if it is the left Kan extension of some $\mathcal{V}$-valued functor on some small category $\mathcal{B}$ along some functor $\mathcal{B} \to \mathcal{A}$.

Example 4.2 When $\mathcal{A}$ is small, every functor $\mathcal{A} \to \mathcal{V}$ is small.

Example 4.3 Taking $\mathcal{V} = \mathbf{Set}$, the constant functor 1 on a large discrete category is not small; nor is the functor Y of Example 4.1.

Example 4.4 For later purposes, let us consider an ordered class A and a subterminal functor $X \colon A^{\mathrm{op}} \to \mathbf{Set}$ (as defined in Example 3.3). Then X is small if and only if there is some small $K \subseteq \operatorname{supp} X$ such that for all $a \in \operatorname{supp} X$, the poset $K \cap {\uparrow} a$ is connected (and in particular, nonempty). This follows from the definition of a small functor as one that is the left Kan extension of its restriction to some small full subcategory.

For arbitrary functors $X, X' \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$, the $\mathcal{V}$-natural transformations $X \to X'$ do not always define an object of $\mathcal{V}$, as Example 4.1 shows in the case $\mathcal{V} = \mathbf{Set}$. But when X is small, they do: it is the (possibly large) end

$$\mathcal{V}\text{-}\mathbf{Nat}(X, X') = \int_a [X(a), X'(a)] \in \mathcal{V}. \tag{9}$$

To see that this end exists, first note that by smallness of X, we can choose a small full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that X is the left Kan extension of its restriction to $\mathcal{C}$. Since $\mathcal{C}$ is small and $\mathcal{V}$ has small limits, the functor $\mathcal{V}$-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ exists, and the universal property of Kan extensions implies that $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]\big(X|_{\mathcal{C}}, X'|_{\mathcal{C}}\big)$ is the end (9).

In particular, the small functors $\mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ form a $\mathcal{V}$-category $\widehat{\mathcal{A}}$. We also write $\check{\mathcal{A}}$ for the *opposite* of the $\mathcal{V}$-category of small functors $\mathcal{A} \to \mathcal{V}$. When $\mathcal{A}$ is small,

$$\widehat{\mathcal{A}} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}], \qquad \check{\mathcal{A}} = [\mathcal{A}, \mathcal{V}]^{\mathrm{op}}.$$

When $\mathcal{A}$ is large, the right-hand sides are in general undefined as $\mathcal{V}$-categories. In the case $\mathcal{V} = \mathbf{Set}$, the right-hand sides can be interpreted as categories that are not locally small, but typically

$$\widehat{\mathcal{A}} \subsetneq [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}], \qquad \check{\mathcal{A}} \subsetneq [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}.$$

A small colimit of small $\mathcal{V}$-valued functors is small, so the $\mathcal{V}$-category $\widehat{\mathcal{A}}$ has small colimits, computed pointwise. Indeed, it is the free cocompletion of $\mathcal{A}$: the Yoneda embedding $\mathcal{A} \hookrightarrow \widehat{\mathcal{A}}$ is the initial functor (in a 2-categorical sense) from $\mathcal{A}$ to a category with small colimits. Dually, $\check{\mathcal{A}}$ is the free completion of $\mathcal{A}$.

Remark 4.5 Isbell used a different size condition, defining a **Set**-valued functor to be **proper** if it admits an epimorphism from a small coproduct of representables ([9], Section 1). (Freyd later called such functors 'petty' [6].) Properness is a weaker condition than smallness, but the universal properties of $\widehat{\mathcal{A}}$ and $\check{\mathcal{A}}$ make smallness a natural choice, and it generalizes smoothly to arbitrary $\mathcal{V}$.

Definition 4.6 A functor $F\colon \mathcal{A} \to \mathcal{B}$ is **representably small** if for each $b \in \mathcal{B}$, the functor

$$N_F(b) = \mathcal{B}(F-, b)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$$

is small, and **corepresentably small** if for each $b \in \mathcal{B}$,

$$N^F(b) = \mathcal{B}(b, F-)\colon \mathcal{A} \to \mathcal{V}$$

is small. (This is dual to the convention in Section 8 of Day and Lack [3].)

Thus, a representably small functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a nerve functor $N_F\colon \mathcal{B} \to \widehat{\mathcal{A}}$, and dually.

Lemma 4.7 *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be functors. If F and G are representably small then so is GF, and dually for corepresentably small.*

Proof Suppose that F and G are representably small, and let $c \in \mathcal{C}$. We must show that $\mathcal{C}(GF-, c)$ is small. By hypothesis, $\mathcal{C}(G-, c)$ is a small colimit of representables, say $\mathcal{C}(G-, c) = W * \mathcal{B}(-, D)$ where $\mathcal{I}$ is a small category, $W\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{V}$ and $D\colon \mathcal{I} \to \mathcal{B}$. Then $\mathcal{C}(GF-, c) = W * \mathcal{B}(F-, D)$, which by hypothesis is a small colimit of small functors, hence small. $\square$

This completes our review of smallness. Now let $\mathcal{A}$ be a category. The **conjugate** of a small functor $X\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is the functor $X^\vee\colon \mathcal{A} \to \mathcal{V}$ defined by

$$X^\vee(a) = \widehat{\mathcal{A}}(X, \mathcal{A}(-, a)).$$

Since $\widehat{\mathcal{A}^{\mathrm{op}}} = \big(\check{\mathcal{A}}\big)^{\mathrm{op}}$, this implies that the conjugate of a small functor $Y\colon \mathcal{A} \to \mathcal{V}$ is the functor $Y^\vee\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ defined by

$$Y^\vee(a) = \check{\mathcal{A}}(Y, \mathcal{A}(a, -)).$$

The conjugate of a small functor need not be small:

Example 4.8 Let $\mathcal{A}$ be a discrete category on a proper class of objects. The small functors $Y\colon \mathcal{A} \to \mathbf{Set}$ are precisely those such that $\operatorname{supp} Y$ is small. So the initial (empty) functor $0\colon \mathcal{A} \to \mathbf{Set}$ is small, but its conjugate is the terminal functor 1, which is not small.

When $\mathcal{V} = \mathbf{Set}$, conjugacy defines functors

$$(\ )^{\vee} : \widehat{\mathcal{A}} \to [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}, \qquad (\ )^{\vee} : \breve{\mathcal{A}} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}],$$

whose codomains are in general not locally small. For a general $\mathcal{V}$ and $\mathcal{A}$, conjugacy is still contravariantly functorial in X and Y, but there are no $\mathcal{V}$-categories $[\mathcal{A}, \mathcal{V}]^{\mathrm{op}}$ and $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ to act as the codomains of $(\ )^{\vee}$. So it no longer makes sense to speak of a conjugacy *adjunction*. However, we do have the following.

Lemma 4.9 *Let $\mathcal{A}$ be a category. Then*

$$\mathcal{V}\text{-}\mathbf{Nat}(X, Y^{\vee}) \cong \mathcal{V}\text{-}\mathbf{Nat}(Y, X^{\vee})$$

naturally in $X \in \widehat{\mathcal{A}}$ and $Y \in \breve{\mathcal{A}}$.

Since X and Y are small, each side of the claimed isomorphism is a well-defined object of $\mathcal{V}$ (equation (9)).

Proof It is routine to verify that each side is naturally isomorphic to $\mathcal{V}\text{-}\mathbf{Nat}(X \boxtimes Y, \mathrm{Hom}_{\mathcal{A}})$, where $\boxtimes$ was defined in (3). □

The isomorphism of Lemma 4.9 gives rise in the usual way to a canonical map $\eta_X : X \to X^{\vee\vee}$ whenever $X : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a small functor such that $X^{\vee}$ is also small. Dually, for any small functor $Y : \mathcal{A} \to \mathcal{V}$ with small conjugate, there is a canonical map $\eta_Y : Y \to Y^{\vee\vee}$.

Remark 4.10 The reuse of the letter η is not an abuse, in that η_X is the same whether X is regarded as a contravariant functor on $\mathcal{A}$ or a covariant functor on $\mathcal{A}^{\mathrm{op}}$. (Compare Remark 2.1.)

In the case $\mathcal{V} = \mathbf{Set}$, the unit transformation η can be described explicitly as follows. Let $X : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ be a small functor with small conjugate. Let $a \in \mathcal{A}$ and $x \in X(a)$. Then $\eta_{X,a}(x) \in X^{\vee\vee}(a)$ is the natural transformation

$$\eta_{X,a}(x) \colon X^{\vee} \to \mathcal{A}(a, -)$$

that evaluates at x: its component at $b \in \mathcal{A}$ is the function

$$\begin{array}{ccc} \widehat{\mathcal{A}}(X, \mathcal{A}(-, b)) & \to & \mathcal{A}(a, b) \\ \xi & \mapsto & \xi_a(x). \end{array}$$

Remark 4.11 Define a category $\mathcal{A}$ to be **gentle** if $\widehat{\mathcal{A}}$ is complete and $\breve{\mathcal{A}}$ is cocomplete. Small categories are certainly gentle. Day and Lack proved that $\widehat{\mathcal{A}}$ is complete if $\mathcal{A}$ is (Corollary 3.9 of [3]), so by duality, any complete and cocomplete category is also gentle. On the other hand, a large discrete category $\mathcal{A}$ is not gentle, as $\widehat{\mathcal{A}}$ has no terminal object.

For a gentle category $\mathcal{A}$, the conjugate of a small functor on $\mathcal{A}$ is again small, so that conjugacy defines a genuine adjunction between $\widehat{\mathcal{A}}$ and $\breve{\mathcal{A}}$. This was shown by Day and Lack in Section 9 of [3].

5 The reflexive completion

The reflexive completion of a category was first defined by Isbell (Section 1 of [9]), for unenriched categories. We consider it for categories enriched in a complete and cocomplete symmetric monoidal closed category $\mathcal{V}$, beginning with the case of small $\mathcal{V}$-categories and then generalizing to arbitrary $\mathcal{V}$-categories. For small categories over $\mathcal{V} = \mathbf{Set}$, our definition is precisely Isbell's. For general categories over $\mathbf{Set}$, there is the set-theoretic difference that our definition uses small functors where his used proper functors (Remark 4.5).

Recall that every adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

between $\mathcal{V}$-categories restricts canonically to an equivalence between full subcategories of $\mathcal{C}$ and $\mathcal{D}$. The subcategory of $\mathcal{C}$ consists of those objects c for which the unit map $c \to GFc$ (in the underlying category of $\mathcal{C}$) is an isomorphism, and dually for $\mathcal{D}$. We call either of these equivalent subcategories the **invariant part** of the adjunction.

The **reflexive completion** $\mathcal{R}(\mathcal{A})$ of a small $\mathcal{V}$-category $\mathcal{A}$ is the invariant part of the conjugacy adjunction

$$\begin{tikzcd} \widehat{\mathcal{A}} \arrow[r, bend left, "\vee"] \arrow[r, phantom, "\perp"] & \breve{\mathcal{A}}. \arrow[l, bend left, "\vee"] \end{tikzcd}$$

When $\mathcal{R}(\mathcal{A})$ is seen as a full subcategory of $\widehat{\mathcal{A}}$, it consists of those functors $X : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ such that the unit map $\eta_X : X \to X^{\vee\vee}$ is an isomorphism; such functors X are called **reflexive**. Dually, $\mathcal{R}(\mathcal{A})$ can be seen as the full subcategory of $\breve{\mathcal{A}}$ consisting of the reflexive functors $\mathcal{A} \to \mathcal{V}$.

Now let $\mathcal{A}$ be an any $\mathcal{V}$-category, not necessarily small. To define reflexivity of a functor X on $\mathcal{A}$, we need $X^{\vee\vee}$ to be defined, so we ask that X and $X^{\vee}$ are small.

Definition 5.1 A functor $X : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is **reflexive** if $X \in \widehat{\mathcal{A}}$, $X^{\vee} \in \breve{\mathcal{A}}$, and the canonical natural transformation $\eta_X : X \to X^{\vee\vee}$ is an isomorphism.

This extends the earlier definition for small $\mathcal{A}$. Although conjugacy for an arbitrary $\mathcal{A}$ does not define an adjunction between $\widehat{\mathcal{A}}$ and $\breve{\mathcal{A}}$, it still induces an equivalence between the full subcategory of $\widehat{\mathcal{A}}$ consisting of the reflexive functors $\mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ and the full subcategory of $\breve{\mathcal{A}}$ consisting of the reflexive functors $\mathcal{A} \to \mathcal{V}$. The **reflexive completion** $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ is either of these equivalent categories.

In the case $\mathcal{V} = \mathbf{Set}$, we have the concrete description of η_X given after Lemma 4.9. It implies that $X \in \widehat{A}$ is reflexive if and only if for each $a \in \mathcal{A}$, every element of $X^{\vee\vee}(a)$ is evaluation at a unique element of $X(a)$.

Remark 5.2 By Remark 4.10, whether a $\mathcal{V}$-valued functor is reflexive does not depend on whether it is considered as a covariant functor on its domain or a contravariant functor on the opposite of its domain. It follows that $\mathcal{R}(\mathcal{A}^{\mathrm{op}}) \simeq \mathcal{R}(\mathcal{A})^{\mathrm{op}}$ for all $\mathcal{V}$-categories $\mathcal{A}$.

Example 5.3 Let $\mathcal{A}$ be a $\mathcal{V}$-category. For each $a \in \mathcal{A}$,

$$\mathcal{A}(-,a)^{\vee} \cong \mathcal{A}(a,-), \qquad \mathcal{A}(a,-)^{\vee} \cong \mathcal{A}(-,a),$$

and the unit map $\mathcal{A}(-,a) \to \mathcal{A}(-,a)^{\vee\vee}$ is an isomorphism. Hence representables are reflexive.

The image of the Yoneda embedding $\mathcal{A} \hookrightarrow \widehat{\mathcal{A}}$ therefore lies in $\mathcal{R}(\mathcal{A})$, when the latter is seen as a subcategory of $\widehat{\mathcal{A}}$. A dual statement holds for $\check{\mathcal{A}}$. There is, then, an unambiguous Yoneda embeddding

$$J_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{R}(\mathcal{A})$$

such that the diagram of full and faithful functors

$$\begin{tikzcd}
 & & \widehat{\mathcal{A}} \\
\mathcal{A} \arrow[urr, hook] \arrow[r, hook, "J_{\mathcal{A}}"] \arrow[drr, hook] & \mathcal{R}(\mathcal{A}) \arrow[ur, hook] \arrow[dr, hook] & \\
 & & \check{\mathcal{A}}
\end{tikzcd} \tag{10}$$

commutes.

6 Examples of reflexive completion

We begin with unenriched examples.

Example 6.1 Let $\mathbf{0}$ denote the empty category. Then $[\mathbf{0}^{\mathrm{op}}, \mathbf{Set}]$ and $[\mathbf{0}, \mathbf{Set}]^{\mathrm{op}}$ are both the terminal category $\mathbf{1}$, so $\mathcal{R}(\mathbf{0}) \simeq \mathbf{1}$. In particular, the reflexive completion of a category need not be equivalent to its Cauchy completion.

Example 6.2 The conjugacy adjunction for the terminal category $\mathbf{1}$ consists of the functors $\mathbf{Set} \rightleftarrows \mathbf{Set}^{\mathrm{op}}$ with constant value 1, giving $\mathcal{R}(\mathbf{1}) \simeq \mathbf{1}$.

Example 6.3 Let $\mathcal{A}$ be a small discrete category with at least two objects. By Example 3.1, for $Y \colon \mathcal{A} \to \mathbf{Set}$,

$$Y^{\vee\vee} \cong \begin{cases} 0 & \text{if } Y \cong 0 \\ \mathcal{A}(a,-) & \text{if } \operatorname{supp} Y = \{a\} \\ 1 & \text{otherwise.} \end{cases}$$

Hence the reflexive functors $\mathcal{A} \to \mathbf{Set}$ are those that are initial, terminal or representable. (Contrast this with Example 6.2, in which the initial functor $\mathbf{1} \to \mathbf{Set}$ is not reflexive.) It follows that $\mathcal{R}(\mathcal{A})$ is $\mathcal{A}$ with initial and terminal objects adjoined.

Example 6.4 Now let $\mathcal{A}$ be a large discrete category. As observed in Example 4.8, the conjugate of the small functor $0 \colon \mathcal{A} \to \mathbf{Set}$ is the non-small functor $1 \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. Hence neither 0 nor 1 is reflexive. The same argument as in Example 6.3 then shows that the only reflexive functors on $\mathcal{A}$ are the representables. Thus, unlike in the small case, the reflexive completion of a large discrete category is itself.

Example 6.5 Let G be a group, regarded as a one-object category. If G is trivial then $\mathcal{R}(G) \simeq \mathbf{1}$ by Example 6.2. Suppose not.

Let X be a right G-set. In Example 3.2, we showed that

$$X^\vee \cong \begin{cases} G_\ell^{X/G} & \text{if } X \text{ is free} \\ \varnothing & \text{otherwise,} \end{cases}$$

and of course a similar result holds for left G-sets. If X is not free then $X^{\vee\vee} = 1$, so the only non-free reflexive G-set is the terminal G-set 1.

Now suppose that X is free. If X is empty then $X^{\vee\vee} = 1^\vee = \varnothing$ (using the nontriviality of G in the second equality), so X is reflexive. Assume now that X is nonempty, write $S = X/G$, and choose $s_0 \in S$. The left G-action on $X^\vee \cong G_\ell^S$ is free, and each orbit contains exactly one element whose s_0-component is the identity element of G, so $X^\vee$ has $|G|^{|S \setminus \{s_0\}|}$ orbits. Writing $|S|-1$ for $|S \setminus \{s_0\}|$, we conclude that $X^\vee$ is a free G-set with

$$|X^\vee/G| = |G|^{|S|-1}.$$

Repeating the argument in the dual situation then gives

$$|X^{\vee\vee}/G| = |G|^{|G|^{|S|-1}-1}.$$

Hence if X is reflexive,

$$|S| = |G|^{|G|^{|S|-1}-1}.$$

By elementary arguments, this implies that $|S| = 1$ (in which case X is representable) or $|G| = |S| = 2$. Hence when $|G| > 2$, the only reflexive right G-sets are $\varnothing$, 1 and G_r.

The remaining case is where G is the two-element group and X is the free G-set on two generators. A direct calculation shows that

$$G_r + G_r \cong G_r \times G_r$$

in $[G^{\mathrm{op}}, \mathbf{Set}]$. Since G is abelian, the same is true for G_ℓ. Now using the adjoint property of conjugates,

$$X^\vee \cong (G_r + G_r)^\vee \cong G_\ell \times G_\ell \cong G_\ell + G_\ell,$$
$$X^{\vee\vee} \cong (G_\ell + G_\ell)^\vee \cong G_r \times G_r \cong G_r + G_r,$$

giving $X^{\vee\vee} \cong X$. We claim that X is reflexive, that is, the unit map $\eta_X \colon X \to X^{\vee\vee}$ is an isomorphism. One of the triangle identities for the conjugacy adjunction implies that $\eta_{W^\vee}$ is split monic for any $W \colon G \to \mathbf{Set}$. But $X \cong (X^\vee)^\vee$, so η_X is an injection between finite sets of the same cardinality, hence bijective, hence an isomorphism.

In summary, the reflexive completion of a group G is as follows:

- if $|G| = 1$ then $\mathcal{R}(G) \cong G$;
- if $|G| = 2$ then $\mathcal{R}(G)$ is the full subcategory of the category of G-sets consisting of the initial G-set, the terminal G-set, the representable G-set G_r, and the four-element G-set $G_r + G_r \cong G_r \times G_r$.

- if $|G| > 2$ then $\mathcal{R}(G)$ is the full subcategory of the category of right G-sets consisting of the initial G-set, the terminal G-set, and the representable G-set. It is equivalent to G with initial and terminal objects adjoined.

Remark 6.6 For a **Set**-valued functor X, the sequence $X, X^\vee, X^{\vee\vee}, X^{\vee\vee\vee}, \ldots$ need not ever repeat itself. For consider functors on the three-element group C_3. By Example 6.5, the conjugate of the free C_3-set $n \times C_3$ on n elements is $3^{n-1} \times C_3$. Since $3^{n-1} > n$ for all $n \geq 2$, no two of the iterated conjugates of $2 \times C_3$ are isomorphic.

Example 6.7 Let M be the two-element commutative monoid whose non-identity element e is idempotent. A covariant or contravariant functor from M to **Set** amounts to a set X together with an idempotent endomorphism f. The representable such functor corresponds to the set $M = \{\mathrm{id}, e\}$ together with the endomorphism with constant value e.

Given a pair (X, f), write $X_0 = \mathrm{im}\, f$ and $X_1 = X \setminus X_0$. A routine calculation shows that

$$\left|\left(X^{\vee\vee}\right)_0\right| = 1 \qquad \text{and} \qquad \left|\left(X^{\vee\vee}\right)_1\right| = 2^{2^{|X_1|}-1} - 1.$$

So if X is reflexive then $|X_0| = 1$ and $|X_1| = 2^{2^{|X_1|}-1} - 1$, and the latter equation implies that $|X_1|$ is 0 or 1. Thus, X is either the representable functor or the terminal functor on M. Both are indeed reflexive.

The reflexive completion of M is, therefore, the full subcategory of the category of M-sets consisting of M itself and 1. This is the free category on a split epimorphism, and is the same as the Cauchy completion of M.

Example 6.8 There is a 7-element monoid whose reflexive completion is large. In particular, the reflexive completion of a finite category need not even be small. This is an example of Isbell to which we return in Example 8.7.

Remark 6.9 The Cauchy completion of a category $\mathcal{A}$ has a well-known concrete description: an object is an object $a \in \mathcal{A}$ together with an idempotent e on a, a map $(a, e) \to (a', e')$ is a map $f : a \to a'$ in $\mathcal{A}$ such that $e'fe = f$, composition is as in $\mathcal{A}$, and the identity on (a, e) is e. It follows that the Cauchy completion of a finite or small category is finite or small, respectively. Example 6.8 implies that the reflexive completion can have no very similar description.

Example 6.10 Let A be a poset, regarded as a category. The conjugacy adjunction of A, when restricted to subterminal functors (Example 3.3), is an adjunction

$$\{\text{downwards closed subsets of } A\} \underset{\downarrow}{\overset{\uparrow}{\rightleftarrows}} \{\text{upwards closed subsets of } A\}^{\mathrm{op}}.$$

Here both sets are ordered by inclusion, and $\uparrow S$ and $\downarrow S$ are the sets of upper and lower bounds of a subset $S \subseteq A$, respectively. The adjointness states that $X \subseteq \downarrow Y \iff Y \subseteq \uparrow X$.

A reflexive functor $A^{\mathrm{op}} \to \mathbf{Set}$ amounts to a downwards closed set $X \subseteq A$ such that $X = \downarrow\uparrow X$. The reflexive completion $\mathcal{R}(A)$ is the set of such subsets X, ordered by inclusion. This is the **Dedekind–MacNeille completion** of the poset A (Definition 7.38 of Davey and Priestley [2]). For example, the reflexive completion of $(\mathbb{Q}, \leq)$ is the extended real line $[-\infty, \infty]$.

The Dedekind–MacNeille completion of a poset is always a complete lattice. (And conversely, any complete lattice is the Dedekind–MacNeille completion of itself.) However, the reflexive completion of an arbitrary small category is far from complete, as Theorem 11.6 shows.

Example 6.11 Now let A be a poset, but regarded as a category enriched in $\mathbf{2}$. By Example 3.4 and the same argument as in Example 6.10, the reflexive completion of A as a $\mathbf{2}$-category is again its Dedekind–MacNeille completion.

Example 6.12 As in Example 3.5, let $\mathcal{V} = \mathbf{Ab}$, let R be a ring, and let M be a right R-module. The double conjugate of M is its double dual

$$M^{\vee\vee} = {}_R\mathbf{Mod}(\mathbf{Mod}_R(M, R_r), R_\ell),$$

so the notion of reflexive functor on R coincides with the algebraists' notion of reflexive module (as in Section 5.1.7 of McConnell and Robson [18]). Hence the reflexive completion of a ring, viewed as a one-object $\mathbf{Ab}$-category, is its category of reflexive modules. In particular, the reflexive completion of a field k is the category of finite-dimensional k-vector spaces.

The rest of this section concerns the case $\mathcal{V} = [0, \infty]$ (Example 3.6), summarizing results from Willerton's analysis [21] of the reflexive completion of a generalized metric space A. It follows from Example 3.6 that $\mathcal{R}(A)$ is the set of distance-decreasing functions $f\colon A^{\mathrm{op}} \to [0, \infty]$ such that

$$f(c) = \sup_{b \in A} \inf_{a \in A} \Bigl(d(c, b) \mathbin{\dot{-}} \bigl(d(a, b) \mathbin{\dot{-}} f(a)\bigr)\Bigr)$$

for all $c \in A$, with metric

$$d(f, g) = \sup_a \bigl(g(a) \mathbin{\dot{-}} f(a)\bigr). \tag{11}$$

The reflexive completion of A consists of the $[0, \infty]$-valued functors equal to their *double* conjugate. But when A is symmetric, covariant and contravariant functors on A can be identified, so we can form the set

$$T(A) = \{\text{distance-decreasing functions } f\colon A \to [0, \infty] \text{ such that } f^\vee = f\}$$

of $[0, \infty]$-valued functors equal to their *single* conjugate. Then $T(A)$, metrized as in equation (11), is called the **tight span** of A. Evidently $A \subseteq T(A) \subseteq \mathcal{R}(A)$. Both inclusions can be strict, as the following example shows.

Example 6.13 Take the symmetric metric space $\{0, D\}$ consisting of two points distance D apart (Figure 2). It follows from the description above that $\mathcal{R}(\{0, D\})$ is the set $[0, D]^2$ with metric

$$d\bigl((t_1, t_2), (u_1, u_2)\bigr) = \max\{u_1 \mathbin{\dot{-}} t_1, u_2 \mathbin{\dot{-}} t_2\}$$

(Example 3.2.1 of [21]). The Yoneda embedding $\{0, D\} \to [0, D]^2$ is given by $0 \mapsto (0, D)$ and $D \mapsto (D, 0)$. The tight span $T(\{0, D\})$ is the interval $[0, D]$ with its usual metric, and embeds in $\mathcal{R}(\{0, D\})$ as shown.

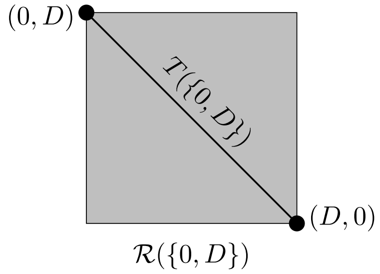

Figure 2: The symmetric metric space $\{0, D\}$ embedded into its tight span $T(\{0, D\})$ and reflexive completion $\mathcal{R}(\{0, D\})$ (Example 6.13).

The tight span construction has been discovered independently several times, as recounted in the introduction of [21]. That the form given here is equivalent to other forms of the definition was established by Dress (Section 1 of [5]). The first to discover it was Isbell [10], who called it the 'injective envelope', $T(A)$ being the smallest injective metric space containing A. But Isbell does not seem to have noticed the connection with Isbell conjugacy.

The tight span is only defined for symmetric spaces, and is itself symmetric (not quite trivially). On the other hand, the reflexive completion of a symmetric space need not be symmetric, as the two-point example shows. Theorem 4.1.1 of [21] states that the tight span is the symmetric part of the reflexive completion:

Theorem 6.14 (Willerton) *Let A be a symmetric metric space. Then the tight span $T(A)$ is the largest symmetric subspace of $\mathcal{R}(A)$ containing A.*

Here 'largest' is with respect to inclusion. A nontrivial corollary is that $\mathcal{R}(A)$ *has* a largest symmetric subspace containing A.

Finally, reflexive completion of metric spaces has arisen in fields far from category theory. Pursuing a project in combinatorial optimization, Hirai and Koichi [8] defined the 'directed tight span' of a generalized metric space. As Willerton showed (Theorem 4.2.1 of [21]), it is exactly the reflexive completion.

7 Dense and adequate functors

Here we gather results on dense and adequate functors that will be used later to characterize the reflexive completion. Some can be found in Isbell's or Ulmer's foundational papers [9, 20] or in Chapter 5 of Kelly [13], while some appear to be new. For the rest of this work, we restrict to unenriched categories, although much of what we do can be extended to the enriched setting.

Definition 7.1 A functor $F \colon \mathcal{A} \to \mathcal{B}$ is **dense** if its nerve functor $N_F \colon \mathcal{B} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ (Definition 4.6) is full and faithful, and **codense** if $N^F \colon \mathcal{B} \to [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$ is full and faithful.

A functor is **small-dense** if dense and representably small, and **small-codense** if codense and corepresentably small.

Remark 7.2 It is a curious fact (not needed here) that while F is dense if and only if N_F is full and faithful, F is full and faithful if and only if N_F is dense.

Example 7.3 Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor with a right adjoint G. It is very well known that G is full and faithful if and only if the counit of the adjunction is an isomorphism. Less well known, but already pointed out by Ulmer in 1968 (Theorem 1.13 of [20]), is that these conditions are also equivalent to F being dense. Thus, any functor F with a full and faithful right adjoint is dense. Indeed, F is small-dense, since $\mathcal{B}(F-, b)$ is representable for each $b \in \mathcal{B}$.

A functor $F\colon \mathcal{A} \to \mathcal{B}$ is small-dense when N_F is full, faithful, and takes values in the category $\widehat{\mathcal{A}}$ of small functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. Then $\mathcal{B}$ embeds fully into $\widehat{\mathcal{A}}$. When $\mathcal{A}$ is small, every dense functor $\mathcal{A} \to \mathcal{B}$ is small-dense.

Example 7.4 The archetypal dense functor is the Yoneda embedding $\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, and the archetypal small-dense functor is the Yoneda embedding $\mathcal{A} \hookrightarrow \widehat{\mathcal{A}}$.

A standard result is that $F\colon \mathcal{A} \to \mathcal{B}$ is dense if and only if every object of $\mathcal{B}$ is canonically a colimit of objects of the form Fa; that is, for each $b \in \mathcal{B}$, the canonical cocone on the diagram

$$(F \downarrow b) \xrightarrow{\mathrm{pr}} \mathcal{A} \xrightarrow{F} \mathcal{B}$$

with vertex b is a colimit cocone (Section X.6 of Mac Lane [17]). In terms of coends, this means that

$$b \cong \int^{a} \mathcal{B}(Fa, b) \times Fa.$$

That the Yoneda embedding is dense gives the **density formula**

$$X \cong \int^{a} X(a) \times \mathcal{A}(-, a)$$

for functors $X\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$.

Example 7.5 When B is an ordered set (or class) and $A \subseteq B$ with the induced order, the inclusion $A \hookrightarrow B$ is dense if and only if it is **join-dense**: every element of B is a join of elements of A.

Lemma 7.6 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a small-dense functor. Then for each $b \in \mathcal{B}$, there is a small diagram $(a_i)_{i \in \mathcal{I}}$ in $\mathcal{A}$ such that $b \cong \operatorname{colim}_i Fa_i$.*

Proof Let $b \in B$. Since F is representably small, we can choose a small diagram (a_i) in $\mathcal{A}$ such that $B(F-, b) \cong \operatorname{colim}_i \mathcal{A}(-, a_i)$. Then by density of F and the density formula,

$$b \cong \int^{a} B(Fa, b) \times Fa \cong \int^{a,i} \mathcal{A}(a, a_i) \times Fa \cong \int^{i} Fa_i. \qquad \square$$

We will be especially interested in the (co)density of functors that are full and faithful. Up to equivalence, such functors are inclusions of full subcategories, which are called **(co)dense** or **small-(co)dense subcategories** if the inclusion functor has the corresponding property.

We now state some basic lemmas on full, faithful and dense functors, beginning with one whose proof is immediate from the definitions.

Lemma 7.7 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a full and faithful functor. Then the composite*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{N_F} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$$

is isomorphic to the Yoneda embedding. □

The next lemma follows from the corollary to Proposition 3.2 in Lambek [15], but we include the short proof for completeness.

Lemma 7.8 *Every full and faithful dense functor preserves all (not just small) limits.*

Proof We use Lemma 7.7. Since N_F is full and faithful, it reflects arbitrary limits; but the Yoneda embedding preserves them, so F does too. □

The composite of two full and faithful dense functors need not be dense. Isbell gave one counterexample (paragraph 1.2 of [9]) and Kelly gave another (Section 5.2 of [13]). Nevertheless:

Lemma 7.9 *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be dense functors, and suppose that G preserves arbitrary colimits. Then GF is dense.*

Proof For $c \in \mathcal{C}$, we have canonical isomorphisms

$$\begin{aligned}
c &\cong \int^{b} \mathcal{C}(Gb, c) \times Gb && (G \text{ is dense}) \\
&\cong \int^{b} \mathcal{C}(Gb, c) \times G\Big(\int^{a} \mathcal{B}(Fa, b) \times Fa\Big) && (F \text{ is dense}) \\
&\cong \int^{a,b} \mathcal{C}(Gb, c) \times \mathcal{B}(Fa, b) \times GFa && (G \text{ preserves colimits}) \\
&\cong \int^{a} \mathcal{C}(GFa, c) \times GFa && (\text{density formula}),
\end{aligned}$$

so GF is dense. □

Definition 7.10 A functor is **adequate** if it is full, faithful, dense and codense, and **small-adequate** if it is full, faithful, small-dense and small-codense.

Remark 7.11 Isbell's foundational paper [9] considered adequacy only for full subcategories. Up to equivalence, this amounts to working only with functors that are full and faithful. For him, fullness and faithfulness were implicit assumptions rather than explicit hypotheses. He used 'left/right adequate' for what is now called dense/codense, and 'adequate' for dense and codense. The word 'dense' was introduced later by Ulmer [20], who extended the theory to arbitrary functors.

Example 7.12 Let $f\colon A \to B$ be an order-preserving map between partially ordered classes. It is full and faithful if and only if it is an order-embedding (that is, reflects the order relation), and by Example 7.5, it is adequate if and only if it is also both join-dense and meet-dense. Small-adequacy means that each element of B is a join of some *small* family of elements of $\operatorname{im} f$, and similarly for meets.

Lemma 7.13 *The classes of adequate and small-adequate functors are each closed under composition.*

Isbell proved an analogue of this result for properly adequate functors (statement 1.6 of [9]), using a different argument.

Proof Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be adequate functors. Then G preserves arbitrary colimits by the dual of Lemma 7.8, so GF is dense by Lemma 7.9. Dually, GF is codense. So GF is adequate. If F and G are small-adequate then so is GF, by Lemma 4.7. $\square$

Lemma 7.14 *For adequate functors* $\mathcal{A} \xrightarrow{F} \mathcal{B} \underset{G'}{\overset{G}{\rightrightarrows}} \mathcal{C}$, *if* $GF \cong G'F$ *then* $G \cong G'$.

Proof We prove the stronger result that if F is codense and G and G' are full, faithful and dense then $GF \cong G'F \Rightarrow G \cong G'$. Indeed, under these assumptions, Lemma 7.8 implies that G preserves all limits, so by codensity of F,

$$Gb \cong G \int_a [B(b, Fa), Fa] \cong \int_a [B(b, Fa), GFa]$$

naturally in $b \in B$ (where $[-,-]$ denotes a power). The same holds for G', so if $GF \cong G'F$ then $G \cong G'$. $\square$

For full subcategories $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, if $\mathcal{A}$ is dense in $\mathcal{C}$ then both intermediate inclusions are dense. More generally:

Lemma 7.15 *Let* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ *be functors, with* G *full and faithful.*

i. *If* GF *is dense then so are* F *and* G.

ii. *If* GF *is small-dense then so is* F.

That G is dense was asserted without proof in statement 1.1 of Isbell [9].

Proof For (i), to prove that F is dense, note that the composite functor

$$\begin{array}{ccccc} \mathcal{B} & \xrightarrow{G} & \mathcal{C} & \xrightarrow{N_{GF}} & [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \\ b & \longmapsto & Gb & \longmapsto & \mathcal{C}(GF-, Gb) \cong \mathcal{B}(F-, b) \end{array}$$

is isomorphic to N_F. But both G and N_{GF} are full and faithful, so N_F is too.

To prove that G is dense, let $c \in \mathcal{C}$. Then

$$\begin{aligned} c &\cong \int^a \mathcal{C}(GFa, c) \times GFa && (GF \text{ is dense}) \\ &\cong \int^{a,b} \mathcal{C}(Gb, c) \times \mathcal{B}(Fa, b) \times GFa && (\text{density formula}) \\ &\cong \int^b \mathcal{C}(Gb, c) \times \int^a \mathcal{C}(GFa, Gb) \times GFa && (G \text{ is full and faithful}) \\ &\cong \int^b \mathcal{C}(Gb, c) \times Gb && (GF \text{ is dense}) \end{aligned}$$

naturally in c, as required.

For (ii), suppose that GF is small-dense. We must prove that F is representably small. Let $b \in \mathcal{B}$. Since G is full and faithful, $\mathcal{B}(F-, b) \cong \mathcal{C}(GF-, Gb)$, which is small since GF is small-dense. $\square$

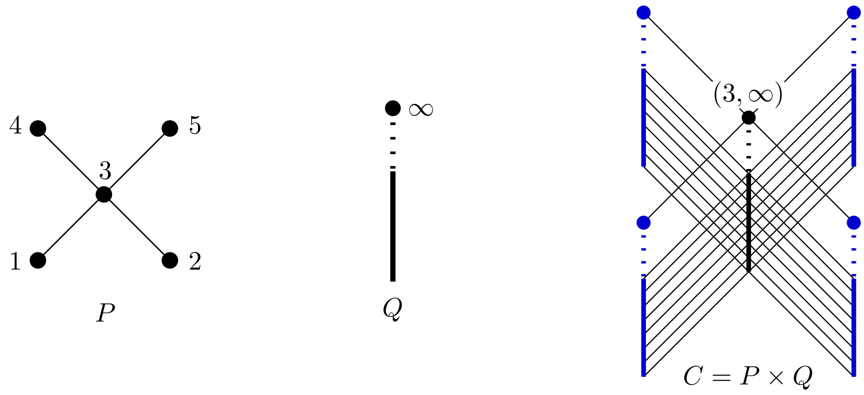

Figure 3: The ordered classes of Example 7.19, with $A \subseteq C$ shown in blue.

Proposition 7.16 *For every category $\mathcal{A}$, the Yoneda embedding $J_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{R}(\mathcal{A})$ is small-adequate.*

Proof We refer to diagram (10) (p. 13). Certainly $J_{\mathcal{A}}$ is full and faithful. Lemma 7.15(ii) applied to $\mathcal{A} \xrightarrow{J_{\mathcal{A}}} \mathcal{R}(\mathcal{A}) \hookrightarrow \widehat{\mathcal{A}}$ implies that $J_{\mathcal{A}}$ is small-dense. By duality, it is also small-codense. □

Lemma 7.15 and its dual immediately imply:

Proposition 7.17 *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be functors, with G full and faithful. If GF is small-adequate then G is adequate and F is small-adequate.* □

Propositions 7.16 and 7.17 have the following corollary. It is implicit in Section 1 of Isbell [9], modulo the difference in size conditions (Remark 4.5).

Corollary 7.18 (Isbell) *Let $\mathcal{A}$ be a category. For any full subcategory $\mathcal{B}$ of $\mathcal{R}(\mathcal{A})$ containing the representables, the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ is small-adequate.* □

Proposition 7.17 does not conclude that G must be *small*-adequate. Indeed, it need not be. This apparently technical point becomes important later, so we give both a counterexample and a sufficient condition for G to be small-adequate.

Example 7.19 We exhibit ordered classes $A \subseteq B \subseteq C$ such that $A \hookrightarrow C$ is small-adequate (hence $A \hookrightarrow B$ is too, by Proposition 7.17) but $B \hookrightarrow C$ is not.

Let P be the 5-element poset shown in Figure 3, and let Q be the ordered class of ordinals with a greatest element ∞ adjoined. Put

$$C = P \times Q, \quad B = C \setminus \{(3, \infty)\}, \quad A = C \setminus (\{3\} \times Q),$$

with the product order on C and the induced orders on $A, B \subseteq C$. We will prove that $A \hookrightarrow C$ is small-adequate but $B \hookrightarrow C$ is not representably small, and, therefore, not small-adequate.

First we show that $A \hookrightarrow C$ is dense, that is, A is join-dense in C (Example 7.5). Let $c \in C$. If $c \in A$ then c is trivially a join of elements of A. Otherwise, $c = (3, q)$ for some $q \in Q$, and then $c = (1, q) \vee (2, q)$ with $(1, q), (2, q) \in A$.

Second, $A \hookrightarrow C$ is codense by the same argument, because it used nothing about the ordered class Q and because $P \cong P^{\mathrm{op}}$.

Third, we show that $A \hookrightarrow C$ is representably small; that is, for each $c \in C$, the subterminal functor $A^{\mathrm{op}} \to \mathbf{Set}$ with support $A \cap {\downarrow} c$ is small. Let $c \in C$. By Example 4.4, we must find a small $K \subseteq A \cap {\downarrow} c$ such that for all $a \in A$, the poset $K \cap {\uparrow} a$ is connected.

If $c \in A$, we can take $K = \{a\}$. Otherwise, $c = (3, q)$ for some $q \in Q$. Put $K = \{(1, q), (2, q)\}$. Given $a \in A \cap {\downarrow} c$, we may suppose without loss of generality that $a = (1, q')$ for some $q' \leq q$, and then $K \cap {\uparrow} a$ is the connected poset $\{(1, q)\}$.

Fourth, $A \hookrightarrow C$ is corepresentably small by the same argument, again because it used nothing about Q and because $P \cong P^{\mathrm{op}}$.

Finally, we show that $B \hookrightarrow C$ is not representably small. In fact, we show that the subterminal functor $B^{\mathrm{op}} \to \mathbf{Set}$ with support $B \cap {\downarrow}(3, \infty)$ is not small. Suppose for a contradiction that it is small. Then we can choose a small class $K \subseteq B \cap {\downarrow}(3, \infty)$ such that for all $b \in B$, the poset $K \cap {\uparrow} b$ is connected. In particular, every element of B is less than or equal to some element of K. Now for each ordinal q we have $(3, q) \in B$, so $(3, q) \leq k$ for some $k \in K$, and then $k = (3, q')$ for some ordinal q' with $q \leq q'$. So if we put $Q' = \{\text{ordinals } q' : (3, q') \in K\}$ then Q' is small (since K is) and every ordinal is less than or equal to some element of Q'. But there is no set of ordinals with this property, a contradiction.

The following companion to Proposition 7.17 uses the notion of gentle category from Remark 4.11.

Lemma 7.20 *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be functors, with G full and faithful. Suppose that $\mathcal{B}$ is gentle. If GF is small-adequate then so is G.*

Proof If GF is adequate then G is adequate by Proposition 7.17, so it only remains to prove that G is representably and corepresentably small. By duality, it is enough to show that G is representably small.

Let $c \in C$. Since GF is small-codense, the dual of Lemma 7.6 implies that $c = \lim_i GFa_i$ for some small diagram (a_i) in $\mathcal{A}$. Then

$$\mathcal{C}(G-, c) \cong \lim_i \mathcal{C}(G-, GFa_i) \cong \lim_i \mathcal{B}(-, Fa_i)$$

as G is full and faithful. Hence $\mathcal{C}(G-, c)$ is a small limit in $[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ of representables.

Since $\mathcal{B}$ is gentle, the subcategory $\widehat{\mathcal{B}}$ of $[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is complete. Limits in $\widehat{\mathcal{B}}$ are computed pointwise because it contains the representables (as noted by Day and Lack in Section 3 of [3]). Hence $\widehat{\mathcal{B}}$ is closed under small limits in $[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$, and in particular, $\mathcal{C}(G-, c)$ is small. □

8 Characterization of the reflexive completion

Here we prove a theorem characterizing the reflexive completion of a category uniquely up to equivalence. It is a refinement and variant of Theorem 1.8 of Isbell [9]. Roughly put, the result is that the reflexive completion of a category $\mathcal{A}$ is the largest category into which $\mathcal{A}$ embeds as a small-adequate subcategory.

This is formally similar to the fact that the completion of a metric space A is the largest metric space in which A is dense.

Lemma 8.1 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a full and faithful small-dense functor. Then $\mathcal{B}(F-,b)^\vee \cong \mathcal{B}(b,F-)$ naturally in $b \in \mathcal{B}$; that is, the diagram*

$$\begin{tikzcd} & \widehat{\mathcal{A}} \arrow[d, "\vee"] \\ \mathcal{B} \arrow[ur, "N_F"] \arrow[r, "N^F"'] & {[\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}} \end{tikzcd}$$

commutes up to natural isomorphism.

The smallness hypothesis guarantees that $\mathcal{B}(F-,b)^\vee$ is defined.

Proof By the hypotheses on F,

$$\begin{aligned} \mathcal{B}(F-,b)^\vee(a) &\cong \widehat{\mathcal{A}}(\mathcal{B}(F-,b), \mathcal{A}(-,a)) \\ &\cong \widehat{\mathcal{A}}(\mathcal{B}(F-,b), \mathcal{B}(F-,Fa)) \\ &\cong \mathcal{B}(b, Fa) \end{aligned}$$

naturally in $a \in \mathcal{A}$ and $b \in \mathcal{B}$. $\square$

For a representably small functor $F\colon \mathcal{A} \to \mathcal{B}$, the nerve functor N_F has image in $\widehat{\mathcal{A}}$. When does it have image in $\mathcal{R}(\mathcal{A})$? The next result provides an answer (given without proof as statement 1.5 of [9]).

Proposition 8.2 (Isbell) *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a full and faithful small-dense functor. Then $\mathcal{B}(F-,b)$ is reflexive for each $b \in \mathcal{B}$ if and only if F is small-adequate.*

Proof Suppose that $\mathcal{B}(F-,b)$ is reflexive for each $b \in \mathcal{B}$. Then we have functors

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{N_F} \mathcal{R}(\mathcal{A})$$

whose composite is the Yoneda embedding $J_\mathcal{A}\colon \mathcal{A} \to \mathcal{R}(\mathcal{A})$ (Lemma 7.7). By Proposition 7.16, $J_\mathcal{A}$ is small-adequate. Hence by Proposition 7.17, F is small-adequate.

Conversely, suppose that F is small-adequate. Let $b \in \mathcal{B}$. Then by Lemma 8.1 and its dual, there are canonical isomorphisms

$$\mathcal{B}(F-,b)^{\vee\vee} \cong \mathcal{B}(b,F-)^\vee \cong \mathcal{B}(F-,b),$$

and $\mathcal{B}(F-,b)$ is reflexive. $\square$

Corollary 8.3 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a small-adequate functor. Then there is a functor $N(F)\colon \mathcal{B} \to \mathcal{R}(\mathcal{A})$, unique up to isomorphism, such that the diagram*

$$\begin{tikzcd} & & \widehat{\mathcal{A}} \\ \mathcal{B} \arrow[urr, "N_F"] \arrow[drr, "N^F"'] \arrow[r, "N(F)"] & \mathcal{R}(\mathcal{A}) \arrow[ur, hook] \arrow[dr, hook] & \\ & & \check{\mathcal{A}} \end{tikzcd}$$

commutes up to isomorphism. Moreover, $N(F)$ is full and faithful. $\square$

Precomposing this whole diagram with the functor $F\colon \mathcal{A} \to \mathcal{B}$ gives the diagram (10) of Yoneda embeddings, by Lemma 7.7.

The main theorem of this section is as follows.

Theorem 8.4 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a small-adequate functor. Then the functor $N(F)\colon \mathcal{B} \to \mathcal{R}(\mathcal{A})$ is adequate, and the triangle*

$$\begin{tikzcd} \mathcal{B} \arrow[rr, dotted, "N(F)"] & & \mathcal{R}(\mathcal{A}) \\ & \mathcal{A} \arrow[ul, "F"] \arrow[ur, hook, "J_{\mathcal{A}}"'] & \end{tikzcd}$$

commutes up to isomorphism. Moreover, $N(F)$ is the unique full and faithful functor $\mathcal{B} \to \mathcal{R}(\mathcal{A})$ such that the triangle commutes, up to isomorphism.

This result is mostly due to Isbell (Theorem 1.8 of [9]). He proved a version for properly adequate functors (Remark 4.5), but without the conclusion that the functor $\mathcal{B} \to \mathcal{R}(\mathcal{A})$ is adequate or unique.

Proof The triangle commutes by Lemma 7.7, $N(F)$ is full and faithful by Corollary 8.3, and then $N(F)$ is adequate by Propositions 7.16 and 7.17. For uniqueness, the same two propositions prove the adequacy of any full and faithful functor making the triangle commute, and the result follows from Lemma 7.14. $\square$

Theorem 8.4 characterizes the reflexive completion uniquely up to equivalence. Indeed, given a category $\mathcal{A}$, form the 2-category whose objects are small-adequate functors out of $\mathcal{A}$ and whose maps are adequate functors between their codomains making the evident triangle commute. Theorem 8.4 states that its terminal object (in a 2-categorical sense) is the Yoneda embedding $J_{\mathcal{A}}\colon \mathcal{A} \hookrightarrow \mathcal{R}(\mathcal{A})$.

Remark 8.5 Corollary 7.18 and Theorem 8.4 together imply that the categories containing $\mathcal{A}$ as a small-adequate subcategory are, up to equivalence, precisely the full subcategories of $\mathcal{R}(\mathcal{A})$ containing $\mathcal{A}$. When $\mathcal{B}$ is a full subcategory of $\mathcal{R}(\mathcal{A})$ containing $\mathcal{A}$, writing $F\colon \mathcal{A} \hookrightarrow \mathcal{B}$ for the inclusion, the uniqueness part of Theorem 8.4 implies that $N(F)$ is the inclusion $\mathcal{B} \hookrightarrow \mathcal{R}(\mathcal{A})$.

Example 8.6 Let A be a poset, and recall Examples 6.10 and 7.12. Loosely, Theorem 8.4 for A states that its Dedekind–MacNeille completion is the largest ordered class containing A and with the property that every element can be expressed as both a join and a meet of elements of A. For example, any poset containing $\mathbb{Q}$ as a join- and meet-dense full subposet embeds into $[-\infty, \infty]$.

Example 8.7 The following example is due to Isbell (Example 1 of [12]). Let $\mathcal{B}$ be the category of sets and partial bijections, and let $\mathcal{A}$ be the full subcategory consisting of a single two-element set. Thus, $\mathcal{A}$ corresponds to a 7-element monoid. Isbell showed that the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ is adequate. It is small-adequate since $\mathcal{A}$ is small. Hence by Theorem 8.4, there is an adequate functor $\mathcal{B} \to \mathcal{R}(\mathcal{A})$. In particular, there is a full and faithful functor from a large category into $\mathcal{R}(\mathcal{A})$, so $\mathcal{R}(\mathcal{A})$ is large (in the strong sense that it is not equivalent to any small category). This proves the statement in Example 6.8: the reflexive completion of a finite category can be large.

Theorem 8.4 shows that when F is small-adequate, $N(F)$ is adequate. But $N(F)$ need not be *small*-adequate, as the following lemma and example show.

Lemma 8.8 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a small-adequate functor such that $N(F)$ is small-adequate. Then for every full and faithful functor $G\colon \mathcal{B} \to \mathcal{C}$ such that GF is small-adequate, G is also small-adequate.*

For a general small-adequate F, without the hypothesis that $N(F)$ is small-adequate, Proposition 7.17 implies that any such functor G is adequate. So the force of the conclusion is that G is *small*-adequate.

Proof The proof will use the functors in the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[rrr, "N(GF)"] & & & \mathcal{R}(\mathcal{A}) \\
& \mathcal{B} \arrow[ul, "G"] \arrow[urr, "N(F)"] & & \\
& & \mathcal{A} \arrow[ul, "F"] \arrow[uur, "J_{\mathcal{A}}"'] &
\end{tikzcd}$$

Let G be a full and faithful functor such that GF is small-adequate. Then there is an induced adequate functor $N(GF)$ as shown. Also, since GF is adequate, Proposition 7.17 implies that G is adequate. Hence by Lemma 7.13, $N(GF)\circ G$ is adequate. Now by Theorem 8.4, $N(F)$ is the unique adequate functor satisfying $N(F) \circ F = J_{\mathcal{A}}$. Since also $N(GF) \circ G \circ F = J_{\mathcal{A}}$ by definition of $N(GF)$, we have $N(GF) \circ G = N(F)$. The hypothesis that $N(F)$ is small-adequate and Proposition 7.17 then imply that G is small-adequate. $\Box$

Example 8.9 Let F be the inclusion $A \hookrightarrow B$ of Example 7.19. As shown there, the conclusion of Lemma 8.8 is false for F. Hence $N(F)$ is not small-adequate.

So, for full subcategories $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{R}(\mathcal{A})$, it is true that $\mathcal{A} \hookrightarrow \mathcal{B}$ is small-adequate and $\mathcal{B} \hookrightarrow \mathcal{R}(\mathcal{A})$ is adequate, but $\mathcal{B} \hookrightarrow \mathcal{R}(\mathcal{A})$ need not be *small*-adequate.

9 Functoriality of the reflexive completion

The reflexive completion differs from many other completions in that it is only functorial in a very restricted sense. First, there is no way to make it act on *all* functors:

Proposition 9.1 *There is no covariant or contravariant pseudofunctor $\mathcal{Q}$ from* **CAT** *to* **CAT** *such that $\mathcal{Q}(\mathcal{A}) \simeq \mathcal{R}(\mathcal{A})$ for all $\mathcal{A} \in$* **CAT**.

Proof Suppose that there is. Write C_2 for the two-element group, viewed as a one-object category. Then C_2 is a retract of $C_2 \times C_2$, so $\mathcal{Q}(C_2)$ is a retract (up to natural isomorphism) of $\mathcal{Q}(C_2 \times C_2)$. Hence $\mathcal{Q}(C_2)$ has at most as many isomorphism classes of objects as $\mathcal{Q}(C_2 \times C_2)$. But by Example 6.5, $\mathcal{Q}(C_2)$ has four isomorphism classes and $\mathcal{Q}(C_2 \times C_2)$ has three, a contradiction. $\Box$

Second, reflexive completion is functorial in the following sense. For a small-adequate functor $F\colon \mathcal{A} \to \mathcal{B}$, the composite $J_\mathcal{B} \circ F$ is small-adequate by Lemma 7.13, giving a functor

$$\mathcal{R}(F) = N(J_\mathcal{B} \circ F)\colon \mathcal{R}(\mathcal{B}) \to \mathcal{R}(\mathcal{A}).$$

By Theorem 8.4, $\mathcal{R}(F)$ is adequate, and up to isomorphism, it is the unique full and faithful functor such that the diagram

$$\begin{tikzcd} \mathcal{R}(\mathcal{A}) & \mathcal{R}(\mathcal{B}) \arrow[l, "\mathcal{R}(F)"'] \\ \mathcal{A} \arrow[u, hook, "J_\mathcal{A}"] \arrow[r, "F"'] & \mathcal{B} \arrow[u, hook, "J_\mathcal{B}"'] \end{tikzcd} \tag{12}$$

commutes. The uniqueness implies that $\mathcal{R}$ defines a pseudofunctor

$$\mathcal{R}\colon (\text{categories and small-adequate functors})^{\text{op}} \\ \to (\text{categories and adequate functors}).$$

When the reflexive completion of a category is regarded as a subcategory of the **Set**-valued functors on it, $\mathcal{R}(F)$ is simply composition with F:

Lemma 9.2 *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a small-adequate functor. Then the squares*

$$\begin{tikzcd} {[\mathcal{A}^{\text{op}}, \mathbf{Set}]} & {[\mathcal{B}^{\text{op}}, \mathbf{Set}]} \arrow[l, "- \circ F"'] \\ \mathcal{R}(\mathcal{A}) \arrow[u, hook] & \mathcal{R}(\mathcal{B}) \arrow[l, "\mathcal{R}(F)"] \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} {[\mathcal{A}, \mathbf{Set}]^{\text{op}}} & {[\mathcal{B}, \mathbf{Set}]^{\text{op}}} \arrow[l, "- \circ F"'] \\ \mathcal{R}(\mathcal{A}) \arrow[u, hook] & \mathcal{R}(\mathcal{B}) \arrow[l, "\mathcal{R}(F)"] \arrow[u, hook] \end{tikzcd}$$

commute up to isomorphism.

Proof Identify $\mathcal{R}(\mathcal{A})$ with the category of reflexive functors $\mathcal{A}^{\text{op}} \to \mathbf{Set}$, and similarly for $\mathcal{B}$. For $Z \in \mathcal{R}(\mathcal{B})$,

$$(\mathcal{R}(F))(Z) = (N(J_B \circ F))(Z) = \mathcal{R}(B)\big((J_\mathcal{B} \circ F)(-), Z\big),$$

which at $a \in \mathcal{A}$ gives

$$\big((\mathcal{R}(F))(Z)\big)(a) = [B^{\text{op}}, \mathbf{Set}]\big(B(-, Fa), Z\big) \cong Z(Fa).$$

Hence $\mathcal{R}(F) \cong - \circ F$, proving the commutativity of the first square. The second follows by duality. □

For example, when F is the inclusion of a small-adequate subcategory and reflexive completions are viewed as categories of functors, $\mathcal{R}(F)$ is restriction.

The pseudofunctor $\mathcal{R}$ applied to a small-adequate functor F produces a functor $\mathcal{R}(F)$ that is adequate but not always small-adequate:

Theorem 9.3 *Let $F: \mathcal{A} \to \mathcal{B}$ be a small-adequate functor. The following are equivalent:*

i. *$N(F)$ is small-adequate;*

ii. *$\mathcal{R}(F)$ is small-adequate;*

iii. *$\mathcal{R}(F)$ is an equivalence.*

If these conditions hold then $NN(F)$ is defined and pseudo-inverse to $\mathcal{R}(F)$.

These equivalent conditions do not always hold, by Example 8.9.

Proof First, since $\mathcal{R}(F) \circ J_\mathcal{B}$ is an adequate functor satisfying $\mathcal{R}(F) \circ J_\mathcal{B} \circ F \cong J_\mathcal{A}$ (diagram (12)), Theorem 8.4 gives

$$\mathcal{R}(F) \circ J_\mathcal{B} \cong N(F), \tag{13}$$

Trivially, (iii) implies (ii). Now assuming (ii), equation (13) gives (i), since the composite of small-adequate functors is small-adequate.

Finally, assume (i). Then $NN(F)$ is defined, and we show that it is pseudo-inverse to $\mathcal{R}(F)$, proving (iii) and the final assertion. Consider the diagram

$$\begin{tikzcd}
\mathcal{R}(\mathcal{A}) \arrow[r, bend right, "NN(F)"'] & \mathcal{R}(\mathcal{B}) \arrow[l, bend right, "\mathcal{R}(F)"'] \\
\mathcal{A} \arrow[u, hook, "J_\mathcal{A}"] \arrow[r, "F"'] & \mathcal{B}. \arrow[u, hook', "J_\mathcal{B}"'] \arrow[ul, "N(F)"]
\end{tikzcd}$$

The bottom-left triangle and the top-right triangle involving $NN(F)$ commute by Theorem 8.4, the top-right triangle involving $\mathcal{R}(F)$ commutes by equation (13), and the outer square commutes by definition of $\mathcal{R}(F)$. Simple diagram chases then show that

$$\mathcal{R}(F) \circ NN(F) \circ J_\mathcal{A} \cong J_\mathcal{A}, \quad NN(F) \circ \mathcal{R}(F) \circ J_\mathcal{B} \cong J_\mathcal{B}.$$

Hence by Lemma 7.14, $\mathcal{R}(F)$ and $NN(F)$ are mutually pseudo-inverse. □

It follows that reflexive completion is idempotent:

Corollary 9.4 (Isbell) *For every category $\mathcal{A}$, the functors*

$$\begin{tikzcd}
\mathcal{R}(\mathcal{A}) \arrow[r, bend left, "J_{\mathcal{R}(\mathcal{A})}"] & \mathcal{R}\mathcal{R}(\mathcal{A}) \arrow[l, bend left, "\mathcal{R}(J_\mathcal{A})"]
\end{tikzcd}$$

define an equivalence $\mathcal{R}(\mathcal{A}) \simeq \mathcal{R}\mathcal{R}(\mathcal{A})$.

A version of this result appeared as part of Theorem 1.8 of Isbell [9], with a partial proof.

Proof Take $F = J_\mathcal{A}: \mathcal{A} \to \mathcal{R}(\mathcal{A})$ in Theorem 9.3. We have $N(J_\mathcal{A}) = 1_{\mathcal{R}(\mathcal{A})}$, which is certainly small-adequate, so $\mathcal{R}(J_\mathcal{A})$ and $NN(J_\mathcal{A})$ are pseudo-inverse. But $NN(J_\mathcal{A}) = N(1_{\mathcal{R}(\mathcal{A})}) = J_{\mathcal{R}(\mathcal{A})}$. □

Corollary 9.5 *The following conditions on a category $\mathcal{A}$ are equivalent:*

i. $J_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathcal{R}(\mathcal{A})$ *is an equivalence;*

ii. *every reflexive functor* $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ *is representable;*

iii. *every reflexive functor* $\mathcal{A} \to \mathbf{Set}$ *is representable;*

iv. $\mathcal{A} \simeq \mathcal{R}(\mathcal{B})$ *for some category* $\mathcal{B}$.

Proof $J_{\mathcal{A}}$ is always full and faithful, so it is an equivalence just when it is essentially surjective on objects. Hence (i) $\iff$ (ii) $\iff$ (iii) by diagram (10). That (iv) is equivalent to (i) follows from Corollary 9.4. □

A category satisfying the equivalent conditions of Corollary 9.5 is **reflexively complete**. It is a self-dual condition: $\mathcal{A}$ is reflexively complete if and only if $\mathcal{A}^{\mathrm{op}}$ is.

In the introduction to Section 8, we compared reflexive completion to metric completion, drawing an analogy between Theorem 8.4 and the characterization of the completion of a metric space A as the largest metric space in which A is dense. The completion of a metric space A can also be characterized as the smallest complete metric space containing A. However, the reflexive analogue of that characterization is false:

Example 9.6 Let $F \colon \mathcal{A} \to \mathcal{B}$ be a small-adequate functor such that $N(F)$ is not small-adequate, as in Example 8.9. Then by Theorem 9.3, the full and faithful functor $\mathcal{R}(F) \colon \mathcal{R}(\mathcal{B}) \to \mathcal{R}(\mathcal{A})$ is not an equivalence. Its image is a full subcategory of $\mathcal{R}(\mathcal{A})$ that is reflexively complete and contains $\mathcal{A}$, but is strictly smaller than $\mathcal{R}(\mathcal{A})$.

Such examples can be excluded by restricting to categories that are gentle. There, the pseudofunctor $\mathcal{R}$ acts somewhat trivially, in the sense of part (i) of Corollary 9.8 below.

Proposition 9.7 *Let* $F \colon \mathcal{A} \to \mathcal{B}$ *be a small-adequate functor. If* $\mathcal{B}$ *is gentle then* $N(F)$ *is small-adequate.*

Proof Apply Lemma 7.20 with $G = N(F)$, recalling that $N(F) \circ F$ is the small-adequate functor $J_{\mathcal{A}}$. □

Corollary 9.8 *Let* $\mathcal{B}$ *be a gentle category. Then:*

i. *for every category* $\mathcal{A}$ *and small-adequate functor* $F \colon \mathcal{A} \to \mathcal{B}$*, the functor* $\mathcal{R}(F) \colon \mathcal{R}(\mathcal{B}) \to \mathcal{R}(\mathcal{A})$ *is an equivalence;*

ii. $\mathcal{R}(\mathcal{A}) \simeq \mathcal{R}(\mathcal{B})$ *for every small-adequate subcategory* $\mathcal{A} \subseteq \mathcal{B}$.

Proof Part (i) follows from Proposition 9.7 and Theorem 9.3, and part (ii) is then immediate. □

For example, every small-adequate subcategory of a complete and cocomplete category $\mathcal{B}$ has the same reflexive completion as $\mathcal{B}$, which by Proposition 11.11 below is $\mathcal{B}$ itself.

10 Reflexive completion and Cauchy completion

Some formal resemblances are apparent between the reflexive and Cauchy completions. Both are idempotent completions; both commute with the operation of taking opposites; and to the analogy between reflexive and metric completion mentioned in the introduction to Section 8, one can add the fact that metric completion is Cauchy completion in the $[0, \infty]$-enriched setting. On the other hand, the reflexive and Cauchy completions are different, as even the example of the empty category shows (Example 6.1). In this section, we describe the relationship between them.

Proposition 10.1 *Let $\mathcal{A}$ be a category.*

i. In $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, the class of reflexive functors is closed under small absolute colimits.

ii. $\mathcal{R}(\mathcal{A})$ is Cauchy complete.

iii. $\overline{\mathcal{A}} \subseteq \mathcal{R}(\mathcal{A})$, when the Cauchy completion $\overline{\mathcal{A}}$ and reflexive completion $\mathcal{R}(\mathcal{A})$ are viewed as subcategories of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

iv. $\mathcal{R}(\overline{\mathcal{A}}) \simeq \mathcal{R}(\mathcal{A})$.

Proof For (i), let $X = \mathrm{colim}_i\, X_i$ be a small absolute colimit of reflexive functors X_i in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. Each X_i is small, so X is small, and X is the absolute colimit of the X_i in $\widehat{\mathcal{A}}$. Since ${}^\vee \colon \widehat{\mathcal{A}} \to [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$ preserves this colimit, $X^\vee \cong \mathrm{colim}_i\, X_i^\vee$ in $[\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$, again an absolute colimit. Since each X_i is reflexive, each $X_i^\vee$ is small. Now $\check{\mathcal{A}} \subseteq [\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$ is complete, hence Cauchy complete, hence closed under small absolute colimits in $[\mathcal{A}, \mathbf{Set}]^{\mathrm{op}}$. So $X^\vee$ is small and is the absolute colimit of the $X_i^\vee$ in $\check{\mathcal{A}}$. Since ${}^\vee \colon \check{\mathcal{A}} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ preserves this colimit, $X^{\vee\vee} \cong \mathrm{colim}_i\, X_i^{\vee\vee}$. Since each X_i is reflexive, so is X.

This proves (i). Colimits in $\mathcal{R}(\mathcal{A}) \subseteq [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ are computed pointwise, so $\mathcal{R}(\mathcal{A})$ has absolute colimits, proving (ii). And $\overline{\mathcal{A}}$ is the closure of $\mathcal{A} \subseteq [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ under absolute colimits, giving (iii).

For (iv), Corollary 7.18 and (iii) imply that the inclusion $F \colon \mathcal{A} \hookrightarrow \overline{\mathcal{A}}$ is small-adequate. We will prove that the induced adequate functor $N(F) \colon \overline{\mathcal{A}} \to \mathcal{R}(\mathcal{A})$ is representably small. It will follow by duality that $N(F)$ is small-adequate, and so by Theorem 9.3 that $\mathcal{R}(F) \colon \mathcal{R}(\overline{\mathcal{A}}) \to \mathcal{R}(\mathcal{A})$ is an equivalence.

First recall that by the 2-universal property of Cauchy completion, restriction along F is an equivalence

$$[\overline{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\simeq} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]. \tag{14}$$

Its pseudo-inverse is left Kan extension along F, and left Kan extension along any functor preserves smallness, so every $Z \colon \overline{\mathcal{A}}^{\mathrm{op}} \to \mathbf{Set}$ such that $Z|_{\mathcal{A}^{\mathrm{op}}}$ is small is itself small.

To prove that $N(F) \colon \overline{\mathcal{A}} \to \mathcal{R}(\mathcal{A})$ is representably small, we regard $\overline{\mathcal{A}}$ and $\mathcal{R}(\mathcal{A})$ as subcategories of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ and recall that $N(F)$ is then the inclusion $\overline{\mathcal{A}} \hookrightarrow \mathcal{R}(\mathcal{A})$ (Remark 8.5). For each $X \in \mathcal{R}(\mathcal{A})$, the Yoneda lemma gives

$$\mathcal{R}(\mathcal{A})(-, X)|_{\mathcal{A}^{\mathrm{op}}} \cong X,$$

so $\mathcal{R}(\mathcal{A})(-, X)|_{\mathcal{A}^{\mathrm{op}}}$ is small, so $\mathcal{R}(\mathcal{A})(-, X)|_{\overline{\mathcal{A}}^{\mathrm{op}}}$ is small, as required. $\square$

Remark 10.2 There is another, more elementary, proof of (iv). One shows that the equivalence (14) restricts to an equivalence $\widehat{\overline{\mathcal{A}}} \xrightarrow{\simeq} \widehat{\mathcal{A}}$, and dually. Then one shows that the conjugacy operations on $\overline{\mathcal{A}}$ and $\mathcal{A}$ commute with these equivalences. It follows that the equivalence (14) also restricts to an equivalence $\mathcal{R}(\overline{\mathcal{A}}) \xrightarrow{\simeq} \mathcal{R}(\mathcal{A})$.

Proposition 10.1(iv) implies:

Corollary 10.3 *Morita equivalent categories have equivalent reflexive completions.* □

Hence the category $\mathcal{R}(\mathcal{A})$ is determined by the category $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, without knowledge of $\mathcal{A}$ itself.

Conjugacy (as opposed to reflexivity) also plays a role in the theory of Cauchy completions. For vector spaces X and Z, there is a canonical linear map

$$\begin{array}{ccc} X^\vee \otimes Z & \to & \mathbf{Vect}(X, Z) \\ \xi \otimes z & \mapsto & \xi(-) \cdot z, \end{array} \tag{15}$$

where $X^\vee$ is the linear dual of X. Analogously, for a category $\mathcal{A}$ and $X, Z \in \widehat{\mathcal{A}}$, there is a canonical map of sets

$$\kappa_{X,Z} \colon X^\vee \odot Z \to \widehat{\mathcal{A}}(X, Z),$$

to be defined. Here

$$X^\vee \odot Z = \int^a X^\vee(a) \times Z(a),$$

and the coend exists since Z is small. The map $\kappa_{X,Z}$ can be defined concretely by specifying a natural family of functions

$$X^\vee(a) \times Z(a) \to [X(b), Z(b)]$$

$(a, b \in \mathcal{A})$, which are taken to be

$$(\xi, z) \mapsto \Big(x \mapsto \big(Z(\xi_b(x))\big)(z)\Big).$$

Equivalently, $X^\vee \odot Z$ is the composite profunctor

$$\mathbf{1} \overset{Z}{\;-\!\!\!+\!\!\!\rightarrow\;} \mathcal{A} \overset{X^\vee}{\;-\!\!\!+\!\!\!\rightarrow\;} \mathbf{1},$$

and the map $\kappa_{X,Z}$ corresponds under the adjunctions (6) to

$$X \odot X^\vee \odot Z \xrightarrow{\varepsilon_X \odot Z} \mathrm{Hom}_{\mathcal{A}} \odot Z \cong Z,$$

where $\varepsilon_X \colon X \odot X^\vee \to \mathrm{Hom}_{\mathcal{A}}$ is the natural transformation of (7).

Proposition 10.4 *Let $\mathcal{A}$ be a small category. The following conditions on a functor $X \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ are equivalent:*

i. $X \in \overline{\mathcal{A}}$, when $\overline{\mathcal{A}}$ is regarded as a subcategory of $\widehat{\mathcal{A}}$;

ii. $\widehat{\mathcal{A}}(X, -): \widehat{\mathcal{A}} \to \mathbf{Set}$ *preserves small colimits;*

iii. $X : \mathbf{1} \nrightarrow \mathcal{A}$ *has a right adjoint in the bicategory* **Prof***;*

iv. $X : \mathbf{1} \nrightarrow \mathcal{A}$ *has right adjoint* $X^\vee$ *in* **Prof***, with counit* ε_X*;*

v. $\kappa_{X,Z} : X^\vee \odot Z \to \widehat{\mathcal{A}}(X, Z)$ *is a bijection for all* $Z \in \widehat{\mathcal{A}}$*.*

Proof The equivalence of (i)–(iii) is standard, and (iv)⇒(iii) is trivial.

To prove (iii)⇒(iv), suppose that X has a right adjoint Y in **Prof**, with counit β. As in any bicategory, this implies that β exhibits Y as the right Kan lift of $\mathrm{Hom}_\mathcal{A}$ through X. (Here we are using a result more often given in the dual form, involving Kan extensions. See, for instance, Theorem X.7.2 of Mac Lane [17], which is stated for **CAT** but the proof is valid in any bicategory.) But as shown at the end of Section 2, ε_X exhibits $X^\vee$ as the right Kan lift of $\mathrm{Hom}_\mathcal{A}$ through X. Hence $(X^\vee, \varepsilon_X) \cong (Y, \beta)$ and (iv) follows.

Now assume (v). The maps $\kappa_{X,Z}$ are natural in $Z \in \widehat{\mathcal{A}}$, so

$$(X^\vee \odot -) \cong \widehat{\mathcal{A}}(X, -): \widehat{\mathcal{A}} \to \mathbf{Set}.$$

But $X^\vee \odot -$ preserves small colimits by the adjointness relations (6), giving (ii).

Finally, assuming (ii), we prove (v). When Z is representable, the function

$$\kappa_{X,Z} : X^\vee \odot Z \to \widehat{\mathcal{A}}(X, Z)$$

is bijective. But every $Z \in \widehat{\mathcal{A}}$ is a small colimit of representables, and both $X^\vee \odot -$ and $\widehat{\mathcal{A}}(X, -)$ preserve small colimits, so $\kappa_{X,Z}$ is bijective for all Z. □

This result can be generalized to enriched categories. When $\mathcal{A}$ is a one-object **Ab**-category corresponding to a field, $\overline{\mathcal{A}}$ is the category of finite-dimensional vector spaces, the maps $\kappa_{X,Z}$ are as defined in equation (15), and we recover the following fact: a vector space X is finite-dimensional if and only if $\kappa_{X,Z}$ is an isomorphism for all vector spaces Z.

Further results on Isbell conjugacy and Cauchy completion can be found in Sections 6 and 7 of Kelly and Schmitt [14].

11 Limits in reflexive completions

A partially ordered set is complete if and only if it is reflexively complete (Example 6.10). In one direction, we show that, more generally, every complete or cocomplete category is reflexively complete (Proposition 11.11). But the converse is another matter entirely: in general, reflexively complete categories have very few limits. We identify exactly which ones.

Lemma 11.1 *Let* $\mathcal{A}$ *be a category.*

i. The inclusion $J_\mathcal{A} : \mathcal{A} \hookrightarrow \mathcal{R}(\mathcal{A})$ *preserves and reflects both limits and colimits.*

ii. The inclusion $\mathcal{R}(\mathcal{A}) \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ *preserves limits and reflects both limits and colimits.*

The inclusion $\mathcal{R}(\mathcal{A}) \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ does not preserve *co*limits, since $J_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathcal{R}(\mathcal{A})$ preserves colimits but the Yoneda embedding $\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ does not.

Proof The statements on reflection are immediate, since both functors are full and faithful.

For (i), the embedding $J_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathcal{R}(\mathcal{A})$ is adequate by Proposition 7.16, so preserves limits and colimits by Lemma 7.8 and its dual.

For (ii), the composite of $\mathcal{R}(\mathcal{A}) \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ with $J_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathcal{R}(\mathcal{A})$ is the Yoneda embedding, which is dense, so $\mathcal{R}(\mathcal{A}) \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is dense by Lemma 7.15. Since it is also full and faithful, it preserves limits by Lemma 7.8. □

We have already shown that reflexively complete categories are Cauchy complete, that is, have absolute limits and colimits (Proposition 10.1(ii)). The next few results show that the reflexive completion of a *small* category also has initial and terminal objects, but that the (co)limits just mentioned are the only ones that generally exist.

For a small category $\mathcal{A}$ and $b \in \mathcal{A}$, write $\mathrm{Cone}(\mathrm{id}, b)$ for the set of cones from the identity to b (natural transformations from $\mathrm{id}_{\mathcal{A}}$ to the constant endofunctor b). This defines a functor $\mathrm{Cone}(\mathrm{id}, -) \colon \mathcal{A} \to \mathbf{Set}$.

Lemma 11.2 *Let $\mathcal{A}$ be a small category. Then $\mathrm{Cone}(\mathrm{id}, -)^{\vee}$ is the terminal functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$.*

Proof Fixing $a \in \mathcal{A}$, we must show that there is exactly one natural transformation $\alpha \colon \mathrm{Cone}(\mathrm{id}, -) \to \mathcal{A}(a, -)$. There is at least one, since we can define $\alpha_b(p) = p_a$ for each $b \in \mathcal{A}$ and $p \in \mathrm{Cone}(\mathrm{id}, b)$.

To prove uniqueness, let $\beta \colon \mathrm{Cone}(\mathrm{id}, -) \to \mathcal{A}(a, -)$. Let $b \in \mathcal{B}$ and $p \in \mathrm{Cone}(\mathrm{id}, b)$. We must prove that $\beta_b(p) = p_a$.

Since p is a cone, $p_b \circ p_c = p_c$ for all $c \in \mathcal{A}$. So we have an equality of cones $p_b \circ p = p$, and then naturality of β gives $p_b \circ \beta_b(p) = \beta_b(p)$. On the other hand, $p_b \circ \beta_b(p) = p_a$ since p is a cone. Hence $\beta_b(p) = p_a$, as required. □

Proposition 11.3 *The reflexive completion of a small category has initial and terminal objects.*

Proof Let $\mathcal{A}$ be a small category. Write 1 for the terminal functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. Then $1^{\vee} \cong \mathrm{Cone}(\mathrm{id}, -)$, so $1^{\vee\vee} \cong 1$ by Lemma 11.2. Hence 1 is reflexive. It follows that 1 is a terminal object of $\mathcal{R}(\mathcal{A}) \subseteq [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. By duality (Remark 5.2), $\mathcal{R}(\mathcal{A})$ also has an initial object. □

Remark 11.4 When $\mathcal{R}(\mathcal{A})$ is viewed as a subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, its initial object is not in general the initial (empty) functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. The case $\mathcal{A} = \mathbf{1}$ already shows this (Example 6.2).

Proposition 11.3 is false for large categories. The proof fails because the terminal functor need not be small. Any large discrete category $\mathcal{A}$ is a counterexample to the statement, since then $\mathcal{R}(\mathcal{A}) \simeq \mathcal{A}$ (Example 6.4).

To show that reflexive completions have no other (co)limits in general, we use the following lemma. A category $\mathcal{I}$ is an **absolute limit shape** if all $\mathcal{I}$-limits are absolute.

Lemma 11.5 *A small category is an absolute limit shape if and only if it admits a cone on the identity.*

This result is at least implicit in the literature on Cauchy completeness, but since we have been unable to find it stated explicitly, we include a proof.

Proof Let $\mathcal{I}$ be a small category admitting a cone

$$\bigl(k \xrightarrow{u_i} i\bigr)_{i \in \mathcal{I}}$$

on the identity. Take functors $\mathcal{I} \xrightarrow{D} \mathcal{A} \xrightarrow{F} \mathcal{B}$ and a limit cone

$$\bigl(L \xrightarrow{p_i} Di\bigr)_{i \in \mathcal{I}} \tag{16}$$

on D. The cone $(Dk \xrightarrow{Du_i} Di)$ induces a map $Dk \xrightarrow{f} L$ in $\mathcal{A}$. Then $f \circ p_k = 1_L$. To show that $(FL \xrightarrow{Fp_i} FDi)$ is a limit cone on FD, let

$$\bigl(B \xrightarrow{r_i} FDi\bigr)_{i \in \mathcal{I}}$$

be any other cone on FD. We have a map $B \xrightarrow{r_k} FDk \xrightarrow{Ff} FL$, and it is routine to check that this is the unique map of cones from (r_i) to (Fp_i). Hence the limit cone (16) is absolute.

We sketch the proof of the converse, which will not be needed here. Suppose that $\mathcal{I}$-limits are absolute. The limit of the Yoneda embedding $\mathcal{I} \hookrightarrow \widehat{\mathcal{I}}$ is $\mathrm{Cone}(-, \mathrm{id}_{\mathcal{I}})$. By absoluteness, this limit is preserved by $\mathrm{colim}_{\mathcal{I}} \colon \widehat{\mathcal{I}} \to \mathbf{Set}$. It follows that $\mathrm{colim}_{\mathcal{I}}(\mathrm{Cone}(-, \mathrm{id}_{\mathcal{I}})) = 1$, so there exists a cone on $\mathrm{id}_{\mathcal{I}}$. □

Theorem 11.6 *Let $\mathcal{I}$ be a small category. The following are equivalent:*

i. $\mathcal{I}$-limits exist in the reflexive completion of every small category;

ii. $\mathcal{I}$-limits exist in every Cauchy complete category with a terminal object;

iii. $\mathcal{I}$ is empty or an absolute limit shape.

By Remark 5.2, dual results hold for colimits.

Proof Every Cauchy complete category has small absolute limits, so (iii)⇒(ii). Every reflexive completion of a small category is Cauchy complete with a terminal object (Propositions 10.1(ii) and 11.3), so (ii)⇒(i). It remains to prove (i)⇒(iii), which we do by contradiction.

Assume (i), and that $\mathcal{I}$ is neither empty nor an absolute limit shape. By Lemma 11.5, $\mathcal{I}$ admits no cone on the identity.

Let $\mathcal{J}$ be the category obtained from $\mathcal{I}$ by adjoining a new object z and maps $p_i^0, p_i^1 \colon z \to i$ for each $i \in \mathcal{I}$, subject to $u \circ p_i^\varepsilon = p_j^\varepsilon$ for each map $i \xrightarrow{u} j$ in $\mathcal{I}$ and $\varepsilon \in \{0, 1\}$. By assumption, the composite

$$\mathcal{I} \hookrightarrow \mathcal{J} \hookrightarrow \mathcal{R}(\mathcal{J})$$

has a limit, L. Since the inclusion $\mathcal{R}(\mathcal{J}) \hookrightarrow [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]$ preserves limits (Lemma 11.1), L is the limit of the composite

$$\mathcal{I} \hookrightarrow \mathcal{J} \hookrightarrow [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}],$$

and is reflexive. Now for $i \in \mathcal{I}$,

$$L(i) = \lim_{i' \in \mathcal{I}} \mathcal{J}(i, i') = \mathrm{Cone}(i, \mathrm{id}_{\mathcal{I}}) = \varnothing,$$

and

$$L(z) = \lim_{i' \in \mathcal{I}} \mathcal{J}(z, i') \cong \{0, 1\}^{\pi_0 \mathcal{I}},$$

where $\pi_0 \mathcal{I}$ is the set of connected-components of $\mathcal{I}$. Write $S = L(z)$. Since $\mathcal{I}$ is nonempty, $|S| \geq 2$. Then

$$L \cong S \times \mathcal{J}(-, z),$$

so

$$L^\vee \cong \mathcal{J}(z, -)^S.$$

Since L is reflexive, the unit map $\eta_{L,z} \colon L(z) \to L^{\vee\vee}(z)$ is surjective. That is, every natural transformation

$$\alpha \colon \mathcal{J}(z, -)^S \to \mathcal{J}(z, -) \tag{17}$$

is the s-projection for some $s \in S$.

We will derive a contradiction by constructing a transformation (17) not of this form. For $i \in \mathcal{I}$, let α_i be the function

$$\begin{array}{cccc} \alpha_i \colon & \mathcal{J}(z, i)^S & \to & \mathcal{J}(z, i) \\ & \big(p_i^{\varepsilon_s}\big)_{s \in S} & \mapsto & p_i^{\min_s \varepsilon_s}, \end{array}$$

and let α_z be the unique function $\mathcal{J}(z, z)^S \to \mathcal{J}(z, z)$. It is routine to check that α defines a natural transformation (17). Hence α is t-projection for some $t \in S$.

Choose some $i \in \mathcal{I}$, as we may since $\mathcal{I}$ is nonempty. Writing δ for the Kronecker delta and recalling that $|S| \geq 2$,

$$\alpha_i\big(\big(p_i^{\delta_{st}}\big)_{s \in S}\big) = p_i^{\min_s \delta_{st}} = p_i^0.$$

But the t-projection of $\big(p_i^{\delta_{st}}\big)_{s \in S}$ is $p_i^{\delta_{tt}} = p_i^1$, a contradiction. $\square$

Remark 11.7 Theorem 11.6 might suggest the idea that reflexive completion is Cauchy completion followed by the adjoining of initial and terminal objects, and there are examples in Section 6 where this is indeed the case. But the group of order 2 (Example 6.5) shows that this is false in general.

Theorem 11.6 concerns limits in reflexive completions of small categories. Every reflexively complete category is the reflexive completion of some category (Corollary 9.5), but not always of a *small* category. For example, any large discrete category is reflexively complete (Example 6.4), but does not have a terminal object and so cannot be the reflexive completion of a small category. The following corollary is the analogue of Theorem 11.6 for the larger class of reflexively complete categories.

Corollary 11.8 *Let $\mathcal{I}$ be a small category. The following are equivalent:*

i. $\mathcal{I}$-limits exist in every reflexively complete category;

ii. $\mathcal{I}$-limits exist in every Cauchy complete category;

iii. $\mathcal{I}$ is an absolute limit shape.

Proof Certainly (iii)⇒(ii), and (ii)⇒(i) because reflexively complete categories are Cauchy complete. Assuming (i), $\mathcal{I}$ is empty or an absolute limit shape by Theorem 11.6. But any large discrete category is a reflexively complete category with no terminal object, so $\mathcal{I}$ is not empty, proving (iii). □

Every **Set**-valued functor on a small category can be expressed as a small colimit of representables. Not every such functor can be expressed as a small *limit* of representables, since then it would preserve small limits.

Lemma 11.9 *Let $\mathcal{A}$ be a category. A functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ is a small limit of representables if and only if it is the conjugate of some small functor $\mathcal{A} \to \mathbf{Set}$.*

Proof Let $X \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. If $X \cong \lim_i \mathcal{A}(-, a_i)$ for some small diagram (a_i) in $\mathcal{A}$ then X is the conjugate of the small functor $\mathrm{colim}_i\, A(a_i, -)$. Conversely, every small functor $\mathcal{A} \to \mathbf{Set}$ can be expressed as a small colimit of representables, and its conjugate is the corresponding small limit of representables. □

Proposition 11.10 *Every reflexive* **Set***-valued functor preserves small limits.*

Proof By Lemma 11.9, every reflexive functor X is a small limit of representables. But representables preserve small limits, so X does too. □

A reflexive functor need not preserve *co*limits. For example, the unique reflexive functor $\mathbf{1} \to \mathbf{Set}$ is the terminal functor, which does not preserve initial objects.

Moreover, although a reflexive functor $\mathcal{A} \to \mathbf{Set}$ preserves all limits that exist in $\mathcal{A}$, it need not be flat. Indeed, many of the examples in Section 6 are of categories $\mathcal{A}$ where the initial functor $0 \colon \mathcal{A} \to \mathbf{Set}$ is reflexive; but 0 is not flat.

Proposition 11.11 *Every complete or cocomplete category is reflexively complete.*

Proof Let $\mathcal{A}$ be a complete category. By Lemma 11.9, every reflexive functor $X \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ is a small limit of representables. But $\mathcal{A}$ is complete, so X is representable. This proves that every complete category if reflexively complete. Since reflexive completeness is a self-dual condition, the dual result follows. □

12 The Isbell envelope

This short section describes the relationship between two constructions due to Isbell. As well as introducing the reflexive completion of a category $\mathcal{A}$ in 1960 [9], he also defined what is now called the Isbell envelope $\mathcal{I}(\mathcal{A})$ in 1966 (naming it the 'couple category': [11], p. 622). See also Garner [7] for a thorough modern treatment of the Isbell envelope.

We begin with an arbitrary adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd},$$

with unit η and counit ε. We already defined the invariant part $\mathbf{Inv}(F, G)$ (Section 5), which comes with full and faithful inclusion functors

$$\begin{tikzcd} \mathbf{Inv}(F,G) \arrow[r] \arrow[d] & \mathcal{C} \\ \mathcal{D} & \end{tikzcd}$$

The **envelope** $\mathbf{Env}(F, G)$ of the adjunction is the category of quadruples

$$(c \in \mathcal{C},\ d \in \mathcal{D},\ f\colon c \to Gd,\ g\colon Fc \to d)$$

such that f and g are each other's transposes. A map $(c, d, f, g) \to (c', d', f', g')$ in $\mathbf{Env}(F, G)$ is a pair of maps

$$(p\colon c \to c',\ q\colon d \to d')$$

such that $(Gq) \circ f = f' \circ p$, or equivalently, $q \circ g = g' \circ (Fp)$. There are canonical functors

$$\begin{tikzcd} & \mathcal{C} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathbf{Env}(F,G) \end{tikzcd} \tag{18}$$

defined by

$$c \mapsto (c, Fc, \eta_c, 1_{Fc}), \qquad d \mapsto (Gd, d, 1_{Gd}, \varepsilon_d)$$

$(c \in \mathcal{C}, d \in \mathcal{D})$, which are full and faithful.

The invariant part and the envelope are related as follows.

Lemma 12.1 *Let $F \dashv G\colon \mathcal{D} \to \mathcal{C}$ be an adjunction.*

i. The full and faithful functors

$$\begin{tikzcd} \mathbf{Inv}(F,G) \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathbf{Env}(F,G) \end{tikzcd}$$

defined above form a 2-pullback square (in the up-to-isomorphism sense).

ii. The composite functor $\mathbf{Inv}(F, G) \to \mathbf{Env}(F, G)$ defines an equivalence between $\mathbf{Inv}(F, G)$ and the full subcategory of objects (c, d, f, g) of $\mathbf{Env}(F, G)$ such that f and g are isomorphisms.

Proof The proof is a series of elementary checks, omitted here. In outline, an object of the 2-pullback of the two functors into $\mathbf{Env}(F, G)$ consists of objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ together with an isomorphism

$$(c, Fc, \eta_c, 1_{Fc}) \xrightarrow{\cong} (Gd, d, 1_{Gd}, \varepsilon_d)$$

in $\mathbf{Env}(F, G)$, and one verifies that this amounts to an object of $\mathbf{Inv}(F, G)$. $\square$

Remark 12.2 The **Inv** and **Env** constructions can be understood as adjoints. Let **ADJ** be the 2-category defined as follows. Objects are adjunctions $(\mathcal{C}, \mathcal{D}, F, G)$. A map $(\mathcal{C}, \mathcal{D}, F, G) \to (\mathcal{C}', \mathcal{D}', F', G')$ consists of functors and natural transformations

$$\bigl(\mathcal{C} \xrightarrow{P} \mathcal{C}',\ \mathcal{D} \xrightarrow{Q}, \mathcal{D}',\ F'P \xrightarrow{\alpha} QF,\ PG \xrightarrow{\beta} G'Q\bigr)$$

such that α and β are each other's mates. The 2-cells are the evident ones. Let $\mathbf{ADJ}_{\cong}$ be the sub-2-category of **ADJ** consisting of all objects, just those maps (P, Q, α, β) for which α and β are isomorphisms, and all 2-cells between them.

There are 2-functors and 2-adjunctions

$$\begin{tikzcd}
 & \mathbf{ADJ} \arrow[dl, bend right, "\mathbf{Env}"'] \\
\mathbf{CAT} \arrow[ur, hook] \arrow[ur, phantom, "\top"] \arrow[dr, hook] \arrow[dr, phantom, "\perp"'] & \\
 & \mathbf{ADJ}_{\cong} \arrow[ul, bend left, "\mathbf{Inv}"] \arrow[uu, hook]
\end{tikzcd}$$

Here, the embedding of **CAT** into $\mathbf{ADJ}_{\cong}$ associates to a category the identity adjunction on it, and **Inv** is its right 2-adjoint. Similarly, **Env** is the right 2-adjoint of $\mathbf{CAT} \hookrightarrow \mathbf{ADJ}$.

Now consider the conjugacy adjunction $\widehat{\mathcal{A}} \rightleftarrows \breve{\mathcal{A}}$ of a small category $\mathcal{A}$. Its invariant part is $\mathcal{R}(\mathcal{A})$. Its envelope is the **Isbell envelope** $\mathcal{I}(\mathcal{A})$. An object of $\mathcal{I}(\mathcal{A})$ is a quadruple

$$(X\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set},\ Y\colon \mathcal{A} \to \mathbf{Set},\ \phi\colon X \to Y^{\vee},\ \psi\colon Y \to X^{\vee})$$

such that ϕ and ψ correspond to one another under the conjugacy adjunction. By the isomorphisms (4), $\mathcal{I}(\mathcal{A})$ is equivalently the category of triples

$$(X\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set},\ Y\colon \mathcal{A} \to \mathbf{Set},\ \chi\colon X \boxtimes Y \to \mathrm{Hom}_{\mathcal{A}}),$$

with the obvious maps between them (as Isbell observed in Section 1.1 of [11]). With this formulation of $\mathcal{I}(\mathcal{A})$, the canonical functor $\widehat{\mathcal{A}} \to \mathcal{I}(\mathcal{A})$ (as in diagram (18)) maps $X \in \widehat{\mathcal{A}}$ to $(X, X^{\vee}, \varepsilon_X)$, where ε_X is the natural transformation of (7). A dual statement holds for $\breve{\mathcal{A}}$.

Lemma 12.1 immediately implies:

Proposition 12.3 *For a small category $\mathcal{A}$, the canonical full and faithful functors*

$$\begin{tikzcd}
\mathcal{R}(\mathcal{A}) \arrow[r] \arrow[d] & \widehat{\mathcal{A}} \arrow[d] \\
\breve{\mathcal{A}} \arrow[r] & \mathcal{I}(\mathcal{A})
\end{tikzcd}$$

form a 2-pullback square. □

Thus, informally, $\mathcal{R}(\mathcal{A}) = \widehat{\mathcal{A}} \cap \breve{\mathcal{A}}$.

References

[1] M. Barr, J. F. Kennison, and R. Raphael. Isbell duality. *Theory and Applications of Categories*, 20(15):504–542, 2008.

[2] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, 2nd edition, 2002.

[3] B. J. Day and S. Lack. Limits of small categories. *Journal of Pure and Applied Algebra*, 210:651–663, 2007.

[4] I. Di Liberti. Codensity: Isbell duality, pro-objects, compactness and accessibility. *Journal of Pure and Applied Algebra*, 224(10):106379, 2020.

[5] A. W. M. Dress. Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces. *Advances in Mathematics*, 53(3):321–402, 1984.

[6] P. Freyd. Several new concepts: lucid and concordant functors, pre-limits, pre-completeness, the continuous and concordant completions of categories. In *Category Theory, Homology Theory and Their Applications (Proceedings of the Battelle Institute Conference, Seattle, Washington, 1968, Volume 3)*, volume 99 of *Lecture Notes in Mathematics*, pages 196–241. Springer, Berlin, 1969.

[7] R. Garner. The Isbell monad. *Advances in Mathematics*, 274:516–537, 2015.

[8] H. Hirai and S. Koichi. On tight spans for directed distances. *Annals of Combinatorics*, 16(3):543–569, 2012.

[9] J. R. Isbell. Adequate subcategories. *Illinois Journal of Mathematics*, 4:541–552, 1960.

[10] J. R. Isbell. Six theorems about injective metric spaces. *Commentarii Mathematici Helvetici*, 39:65–74, 1964.

[11] J. R. Isbell. Structure of categories. *Bulletin of the American Mathematical Society*, 72:619–655, 1966.

[12] J. R. Isbell. Small adequate subcategories. *Journal of the London Mathematical Society*, 43:242–246, 1968.

[13] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982. Also *Reprints in Theory and Applications of Categories* 10:1–136, 2005.

[14] G. M. Kelly and V. Schmitt. Notes on enriched categories with colimits of some class. *Theory and Applications of Categories*, 14(7):399–423, 2005.

[15] J. Lambek. *Completions of Categories*, volume 24 of *Lecture Notes in Mathematics*. Springer, Berlin, 1966.

[16] F. W. Lawvere. Metric spaces, generalized logic and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, XLIII:135–166, 1973. Also *Reprints in Theory and Applications of Categories* 1:1–37, 2002.

[17] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics 5. Springer, New York, 1971.

[18] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian Rings*. Graduate Studies in Mathematics. American Mathematical Society, Providence, Rhode Island, 2001.

[19] C. McLarty. The rising sea: Grothendieck on simplicity and generality. In *Episodes in the History of Modern Algebra (1800–1950)*, volume 32 of *History of Mathematics*, pages 301–325. American Mathematical Society, Providence, RI, 2007.

[20] F. Ulmer. Properties of dense and relative adjoint functors. *Journal of Algebra*, 8:77–95, 1968.

[21] S. Willerton. Tight spans, Isbell completions and semi-tropical modules. *Theory and Applications of Categories*, 28(22):696–732, 2013.